Advances in Nonlinear Variational Inequalities
ISSN: 1092-910X
Vol 28 No. 2 (2025)

A Computational Approach of Singularly Perturbed Singular Boundary Value Problem using an Adaptive Spline

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Article History:

Received: *27-07-2024*

Revised: *11-09-2024*

Accepted: *18-09-2024*

Abstract:

This paper develops a different approach for the numerical solution of singularly perturbed two-point singular boundary-value problems using an adaptive spline. The difference method is designed for nonstandard finite differences of the first order derivative and adaptive spline. The suggested scheme is accurate to fourth order and works for both singular and non-singular problems. Convergence analysis of a difference scheme is discussed in this paper. The numerical results are given to illustrate the efficiency of the proposed methods in comparison to the existing method in the literature.

Keywords: Singular perturbation, Singularity, Adaptive spline, Root mean square errors.

1. Introduction

Singularly perturbed singular boundary value problems (SPSBVP) are encountered in many areas of engineering and science, including quantum mechanics, fluid mechanics, chemical reactor theory, optimal control, and so on. There are numerous methods for solving the problems listed above. The authors in [1-2] solved similar problems in the past, but their solutions are only applicable to non-singular problems. Mohanty et al. [3] proposed a spline in compression method to solve SPSBVP numerically. Mohanty and Jha [4] developed spline in compression methods on variable mesh to solve the same problem. Bava [5] suggested several computational techniques based on spline to solve linear singularly perturbed boundary value problems. Rashidinia et al. [6] solved a singularly perturbed singular boundary value problem using cubic spline.

Phaneendra et al. [7] suggested a fourth order convergence finite difference scheme using a non-polynomial spline for solving SPSBVP. Wang et al. [8] proposed a new numerical scheme using Fourier sine series for the solution of singular and singularly perturbed boundary value problems. The procedure removes the singularity of the problem in a natural way. Miller et al. [9] devised uniformly convergent numerical schemes composed of upwind-difference operators on uniform meshes. Geng [10] constructed a novel method using reproducing kernel method for solving a class of singularly perturbed boundary value problems.

Zahra and Ashraf [11] solved a singularly perturbed semi-linear boundary value problem having two-parameters using an exponential spline on a Shishkin mesh. Rai and Sharma [12] solved a problem having layer behaviour which arise in computational neuroscience in the modelling of neuronal variability. Authors derived an exponentially fitted scheme based on Il'in-Allen-Southwell on a specially designed mesh. Miller and O'Riordan [13] constructed a robust numerical method with a specially constructed piecewise-uniform mesh that is used to solve a singularly perturbed problem arising in the modelling of enzyme kinetics. Liu and Huang [14] proposed tailored finite point method for a fourth order singularly perturbed partial differential equation reduced to a second order PDE system with coupled boundary conditions. Swarnakar et al. [15] applied exponential spline method for numerical solution of singularly perturbed two-point singular boundary value problem.

1. Objectives

The proposed adaptive spline method provides high accuracy across the entire solution domain. It remains numerically stable and convergent, in the presence of the small perturbation perameter. This method is particularly well-suited for solving singularly perturbed two-point singular boundary value problems efficiently.

3. Problem description

Let a singularly perturbed two-point singular boundary value problem

$$\varepsilon \frac{d^2w}{dv^2} + p(v)\frac{dw}{dv} + q(v)w(v) = g(v), \qquad 0 < v < 1 \tag{1}$$

where $0 < \varepsilon << 1, p(v) < L < 0, q(v) > 0$ and L is a positive constant and the boundary conditions are $w(0) = C$ and $w(1) = D$ (2)

where C and D are finite constants.

4. Numerical Approach using an Adaptive Spline

Consider a mesh with grid points v_i on $[a,b]$ such that Δ: $a = v_0 < v_1 <\ldots< v_{N-1} < v_N = b$ where $h = v_i - v_{i-1}$ for $i = 1,2,\ldots,N$. Let the exact solution of w at the grid point v_i denoted as W_i and w_i be its approximate solution.

Let $p_i = p(v_i)$, $q_i = q(v_i)$ and $g_i = g(v_i)$.

A function $S_\Delta(v,\mu)$ of class $C^2[a,b]$ which interpolates $w(v)$ at the mesh points v_i depends on a parameter μ, reduces to cubic spline $S_\Delta(v)$ in $[a,b]$ as $\mu \to 0$ is termed as adaptive spline function. The spline function satisfies the following differential equation.

$$aS_\Delta''(v,\mu) - bS_\Delta'(v,\mu) = (aM_i - bm_i)\left(\frac{v-v_{i-1}}{h}\right) + (aM_{i-1} - bm_{i-1})\left(\frac{v_i - v}{h}\right) \tag{3}$$

where a and b are constants, $S_\Delta'(v_i,\mu) = m_i$, $S_\Delta''(v_i,\mu) = M_i$ and $v \in (v_{i-1}, v_i)$

Solving the Eq. (3) and applying the interpolation conditions $S_\Delta(v_{i-1},\mu) = w_{i-1}$ and $S_\Delta(v_i,\mu) = w_i$, it becomes

$$S_\Delta(v,\mu) = R_i + A_i e^{\mu Z} - \frac{h^2}{\mu^3}\left[\frac{\mu^2 Z^2}{2} + \mu Z + 1\right] + (M_i - \frac{\mu}{h}m_i)\left(\frac{v_i - v}{h}\right)$$
$$+ \frac{h^2}{\mu^3}\left[\frac{\mu^2(1-Z)^2}{2} + \mu(1-Z) + 1\right](M_{i-1} - \frac{\mu}{h}m_{i-1}) \tag{4}$$

where

$$R_i(e^\mu - 1) = -w_i + e^\mu w_{i-1} - \frac{h^2}{\mu^3}\left[\left(\frac{\mu^2}{2} + \mu + 1\right) - \mu e^\mu\right]\left(M_i - \frac{\mu}{h}m_i\right) - \frac{h^2}{\mu^3}\left[\left(\frac{\mu^2}{2} - \mu + 1\right) - \mu\right]\left(M_{i-1} - \frac{\mu}{h}m_{i-1}\right),$$

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ISSN: 1092-910X
Vol 28 No. 2 (2025)

$$A_i(e^{\mu}-1)=w_i-w_{i-1}+\frac{h^2}{\mu^3}\left[\left(\frac{\mu}{2}+1\right)\left(M_i-\frac{\mu}{h}m_i\right)\right]+\left(\frac{\mu}{2}-1\right)\left(M_{i-1}-\frac{\mu}{h}m_{i-1}\right),$$

$\mu=\frac{bh}{a}$ and $Z=\frac{v-v_{i-1}}{h}$

on the interval $[v_i,v_{i+1}]$, the function $S_{\Delta}(v,\mu)$ is obtained by replacing i by $(i+1)$ in the Eq. (4). Using the continuity condition of the first or second derivatives of $S_{\Delta}(v,\mu)$ at $v=v_i$ generates the following equation

$$\left(M_{i+1}-\frac{\mu}{h}m_{i+1}\right)\left[e^{-\mu}\left(\frac{\mu^2}{2}+\mu+1\right)-1\right]+\left(M_i-\frac{\mu}{h}m_i\right)$$
$$\left[e^{-\mu}\left(\frac{\mu^2}{2}-\mu-2\right)+\left(-\frac{\mu^2}{2}-\mu+2\right)+\left(M_{i-1}-\frac{\mu}{h}m_{i-1}\right)\left(e^{\mu}-1-\frac{\mu^2}{2}+\mu\right)\right]$$
$$=-\frac{\vartheta^2}{h^2}\left[e^{-\vartheta}w_{i+1}-\left(1+e^{-\vartheta}\right)w_i+w_{i-1}\right] \quad (5)$$

Some additional relations for the adaptive spline are listed below

(i) $m_{i-1}=-h(R_1M_{i-1}+R_2M_i)+\frac{1}{h}(w_i-w_{i-1})$

(ii) $m_i=h(R_3M_{i-1}+R_4M_i)+\frac{1}{h}(w_i-w_{i-1})$

(iii) $\frac{Xh}{2\mu}M_{i-1}=-(R_4m_{i-1}+R_2m_i)+\frac{A_1}{h}(w_i-w_{i-1})$

(iv) $\frac{Xh}{2\mu}M_i=(R_3m_{i-1}+R_1m_i)+\frac{B_2}{h}(w_i-w_{i-1})$ (6)

where $R_1=\frac{1}{4}(1+X)+\frac{X}{2\mu}$, $R_2=\frac{1}{4}(1-X)-\frac{X}{2\mu}$, $R_3=\frac{1}{4}(1+X)-\frac{X}{2\mu}$,

$R_4=\frac{1}{4}(1-X)+\frac{X}{2\mu}$, $A_1=\frac{1}{2}(1-X)$, $A_2=-\frac{1}{2}(1+X)$ and $X=cot\left(\frac{\mu}{2}\right)-\frac{2}{\mu}$

and obtain,

$$R_2M_{i+1}+(R_1+R_4)M_i+R_3M_{i-1}=\frac{1}{h^2}(w_{i+1}-2w_i+w_{i-1}) \quad (7)$$

in the limiting case $\mu\to 0$, then

$$X=0, \frac{X}{\vartheta}=\frac{1}{6}, R_1=R_4=\frac{1}{3}, R_2=R_3=\frac{1}{6}, A_1=\frac{1}{2}, A_2=-\frac{1}{2}$$

and the spline function given by the Eq. (4) reduces to cubic spline.

5. Application of the method

At the grid points $v_{i,}$ the proposed Eq. (1) may be discretized by

$$\varepsilon w_i''+p_iw_i'+q_iw_i=g_i \quad (8)$$

by using moment of spline in Eq. (8), we get

$$\varepsilon M_i+p_iw_i'+q_iw_i=g_i \quad (9)$$

where $\varepsilon M_i=g_i-p_iw_i'-q_iw_i$

using the following approximations for first derivative of w, let

$$w_{i+1}'\approx\frac{1}{2h}[3w_{i+1}-4w_i+w_{i-1}] \quad (10)$$

$$w_{i-1}' \approx \frac{1}{2h}[-w_{i+1} + 4w_i - 3w_{i-1}] \qquad (11)$$

$$w_i' \approx \frac{1}{2h}[1 + 2\omega h^2 q_{i+1} + \omega h(3p_{i+1} + p_{i-1})]w_{i+1} - \frac{1}{2h}[1 + 2\omega h^2 q_{i-1} - \omega h(p_{i+1} + 3p_{i-1})]w_{i-1} - 2\omega[p_{i+1} + p_{i-1}] + \omega h[g_{i+1} + g_{i-1}] \qquad (12)$$

by substituting the Eqs. (9) and (10) -(12) in the Eq. (7) and simplifying, the following tri-diagonal system is derived.

$$[-\varepsilon - \frac{3}{2}R_3\, p_{i-1}\, h - (R_1 + R_4)\, p_i h\{1 + 2\omega h^2 q_{i-1} - \omega h(p_{i+1} + 3p_{i-1})\} + \frac{R_2}{2} p_{i+1} h + R_3 q_{i-1} h^2]w_{i-1} + [2\varepsilon + 2R_3\, p_{i-1}\, h - 4(R_1 + R_4)\, p_i h^2 \omega(p_{i+1} + p_{i-1}) - 2R_2 p_{i+1} h + 2(R_1 + R_4) q_i h^2]w_i + [-\varepsilon - \frac{R_3}{2}\, p_{i-1}\, h + (R_1 + R_4)\, p_i h\{1 + 2\omega h^2 q_{i-1} + \omega h(3p_{i+1} + p_{i-1})\} + \frac{3}{2} R_2 \omega p_{i+1} h + R_2 q_{i+1} h^2]w_{i+1} \quad \text{for } i = 1,2,\dots,N \qquad (13)$$

solving the Eq. (13), the solution $w(v)$ of are $w_1, w_2, \dots, w_{n-1}$ at $v_1, v_2, \dots, v_{n-1}$.

However, the method fails when the coefficients of $p(v), q(v)$ and $g(v)$ have singularities at v=0. Hence, the scheme in the Eq. (13) fails for i=1. To overcome this, the following approximations can be considered.

$$p_{i\pm1} = p_i \pm hp_i{}' + O(h^2) \qquad (14)$$

$$q_{i\pm1} = q_i \pm hq_i{}' + O(h^2) \qquad (15)$$

$$g_{i\pm1} = g_i \pm hg_i{}' + O(h^2) \qquad (16)$$

now using the approximations from the Eqs. (14) -(16) in the Eq. (13) and ignoring higher order terms, the following difference schemes obtains in compact form as

$$\left[-\varepsilon - \frac{3}{2}R_3(p_i - hp_i{}')h - (R_1 + R_4)p_i h + 2\omega h^2(q_i - hq_i{}') - 2\omega h(2p_i - hp_i{}') + \frac{R_2}{2}(p_i + hp_i{}')h + R_3(q_i - hq_i{}')h^2\right]w_{i-1} + \left[2\varepsilon + 2R_3(p_i - hp_i{}')h - 8(R_1 + R_4)p_i^2 h^2 \omega - 2R_2(p_i + hp_i{}')h + 2(R_1 + R_4)q_i h^2\right]w_i + \left[-\varepsilon - \frac{R_3}{2}(p_i - hp_i{}')h + (R_1 + R_4)p_i h + 2\omega h^2(q_i - hq_i{}') + 2\omega h(2p_i + hp_i{}') + \frac{3}{2}R_2\omega(p_i + hp_i{}')h + R_2(q_i + hq_i{}')h^2\right]w_{i+1}$$

$$= -h^2[(R_1 + R_2 + R_3 + R_4)g_i + R_2 - R_3 - 2(R_1 + R_4)\omega hp_i hg_i{}'] \quad \text{for } i = 1,2,\dots,N \qquad (17)$$

6. Convergence Analysis

To establish the convergence analysis, the difference scheme of the Eq. (17) is being considered as

$$[-\varepsilon - \frac{3}{2}R_3\,(p_i - hp_i')h - (R_1 + R_4)\, p_i h\{1 + 2\omega h^2(q_i - hq_i') - 2\omega h(2p_i - hp_i')\} + \frac{R_2}{2}(p_i + hp_i')h + R_3(q_i - hq_i')h^2]w_{i-1} + [2\varepsilon + 2R_3\,(p_i - hp_i')h - 8(R_1 + R_4)\, p_i^2 h^2 \omega - 2R_2(p_i + hp_i')h + 2(R_1 + R_4)q_i h^2]w_i + [-\varepsilon - \frac{R_3}{2}\,(p_i - hp_i')\, h + (R_1 + R_4)\, p_i h\{1 + 2\omega h^2(q_i - hq_i') + 2\omega h(2p_i + hp_i')\} + \frac{3}{2}R_2\omega(p_i + hp_i')h + +R_2(q_i + hq_i')h^2]w_{i+1} + b_i + T_i(h) = 0 \quad \text{with } w(0) = C, w(1) = D \text{ for } i = 1,2,,\dots,N \qquad (18)$$

here $b_i = h^2[(R_1 + R_2 + R_3 + R_4)g_i + \{R_2 - R_3 - 2(R_1 + R_4)\omega hp_i)hg_i'\}]$ and the truncation error is

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$T_i(h) = \varepsilon[(R_1 + R_2 + R_3 + R_4) - 1]h^2 w_i'' + \Big[\varepsilon(R_2 - R_3)h^3 + \Big(\frac{1}{2}(R_2 + R_3) + 2\omega\varepsilon(R_1 + R_4) - \frac{1}{6}(R_1 + R_2 + R_3 + R_4)p_i h^4\Big)\Big] w_i''' + \frac{\varepsilon}{12}[-1 + 6(R_2 + R_3)]h^4 w_i^{(iv)} + \frac{1}{12}(R_3 - R_2)[p_i w_i^{(iv)} + (2p_i' + q_i)w_i''' + 6(p_i'' + q_i')w_i'' + 2(p_i''' + 3q_i'')w_i' + 2w_i q_i''' - 2g_i''']h^5 + O(h^6)$ for $i = 1,2,\dots,N$.

i.e., $T_i(h) = O(h^6)$ for $R_2 = R_3 = \frac{1}{12}, R_1 + R_4 = \frac{5}{12}$ and $\omega = \frac{-1}{20\varepsilon}$

with the given boundary conditions, $w_0 = C$ and $w_{N+1} = D$. Incorporating the boundary conditions, the matrix form of the system of the Eq. (18) is

$$(D + J)W + F + T(h) = 0 \qquad (19)$$

where $$D = [-\varepsilon, 2\varepsilon, -\varepsilon] = \begin{bmatrix} 2\varepsilon & -\varepsilon & 0 & 0 & \dots & 0 \\ -\varepsilon & 2\varepsilon & -\varepsilon & 0 & \dots & 0 \\ 0 & -\varepsilon & 2\varepsilon & -\varepsilon & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & -\varepsilon & 2\varepsilon \end{bmatrix}$$

$$J = [z_i, k_i, l_i] = \begin{bmatrix} k_1 & l_1 & 0 & 0 & \dots & 0 \\ z_2 & k_2 & l_2 & 0 & \dots & 0 \\ 0 & z_3 & k_3 & l_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & z_{N-1} & k_{N-1} \end{bmatrix}$$

$$z_i = \frac{1}{2}hp_i(3R_3 - R_2) + \frac{1}{2}h^2 p_i'(R_2 + 3R_3) + \frac{1}{2}hp_i(R_1 + R_4)\{1 + 2\omega h^2(q_i - hq_i') - 2\omega h(2q_i - hq_i') - R_3 h^2(q_i - hq_i'),$$

$$k_i = 2h[(R_2 - R_3 p_i) + (R_2 + R_3)hq_i'] + (R_1 + R_4)(4\omega p_i^2 - q_i')h^2,$$

$$l_i = \frac{1}{2}hp_i(R_3 - 3R_2) - \frac{1}{2}h^2 p_i'(3R_2 + R_3) - \frac{1}{2}hp_i(R_1 + R_4)\big[1 + 2\omega h^2\big(q_i + hq_i'\big)h^2 + 2\omega h\big(2p_i + hp_i'\big)\big],$$

$$F = [b_1 + (-\varepsilon + z_1)C, b_2, b_3, \dots, b_{N-1} + (-\varepsilon + w_{n-1})D] \text{ and}$$

$$\tilde{q}_i = h^2 p_i[(R_1 + R_2 + R_3 + R_4)g_i + \{R_2 - R_3 - 2(R_1 + R_4)h\omega p_i\}hg_i'] \text{ respectively}$$

for $i = 1,2,\dots,N-1$.

Let $W = [W_1, W_2, \dots, W_{N-1}]^T$, $T(h) = [T_1, T_2, \dots, T_{N-1}]^T$, $O = [0,0,\dots,0]^T$ are associated vectors with the Eq. (19). Let $w = [w_1, w_2, \dots, w_{N-1}]^T \cong W$ which satisfies the equation

$$(D + J)w + F = 0 \qquad (20)$$

Let $e_i = y_i - Y_i$ for $i = 1,2,\dots,N-1$ be the discretization error so that $e = [e_1, e_2, \dots, e_{N-1}]^T = w - W$. Subtracting the Eq. (19) from the Eq. (20), the error equation is obtained as

$$(D + J)E = T(h) \qquad (21)$$

Let $|p(v)| \le C_1$, $|p'(v)| \le C_2$, $|q(v)| \le C_3$ and $|q'(v)| \le C_4$ which are positive constants. If $J_{i,j}$ be the $(i,j)^{th}$ element of J, then

$$\left|j_{i,i+1}\right| = |l_i| \leq \left|\begin{matrix}\frac{h}{2}[C_1(R_3 - 3R_2) - hC_2(3R_2 + R_3) - C_1(R_1 + R_4)\{1 + 2\omega h^2(C_3 + C_4 h) \\ +2\omega h(2C_1 + hC_2)\}]\end{matrix}\right|$$

for $i = 1,2,\ldots,N-2$

$$\left|j_{i,i-1}\right| = |z_i| \leq \left|\begin{matrix}\frac{h}{2}[C_1(3R_3 - R_2) + hC_2(R_2 + 3R_3) + C_1(R_1 + R_4)\{1 + 2\omega h^2(C_3 + C_4 h) \\ -2\omega h(2C_1 - hC_2)\} - R_3 h^2(C_3 - C_4 h)]\end{matrix}\right|$$

for $i = 2,3,\ldots,N-1$

thus, for small values of h, let

$$\left|j_{i,i+1}\right| < \varepsilon \text{ for } i = 1,2,\ldots,N-2. \quad (22)$$

and

$$\left|j_{i,i-1}\right| < \varepsilon \text{ for } i = 2,3,\ldots,N-1 \quad (23)$$

hence $(D+J)$ is irreducible [15]. Let $\tilde{S}_i$ be the sum of the elements of the i^{th} row of the matrix $(D+J)$, then

$$\tilde{S}_i = \varepsilon + k_i + l_i \text{ for } i = 1,$$

$$\tilde{S}_i = z_i + k_i + l_i \text{ for } i = 2,3,\ldots,N-2,$$

$$\tilde{S}_i = \varepsilon + k_i + l_i \text{ for } i = N-1$$

Let $C_{1^*} = min|p(v)|, C_1^* = max|p(v)|, C_{2^*} = min|p'(v)|, C_2^* = max|p'(v)|,$
$D_{1^*} = min|q(v)|, D_1^* = max|q(v)|, D_{2^*} = min|q'(v)|, D_2^* = max|q'(v)|$ for $i = 1,2,\ldots,N$

Since $0 < \varepsilon \ll 1$ and $\varepsilon\alpha O(h)$, it is verified that $(D+J)$ is monotone [16,17],

hence$(D+J)^{-1}$exhist and $(D+J)^{-1} \geq 0$. Thus, from the Eq. (21), it is clear that

$$\|E\| \leq \left\|(D+J)^{-1}\right\|\|T(h)\| \quad (24)$$

let $(D+J)^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(D+J)^{-1}$ and define

$$\left\|(D+J)^{-1}\right\| = max\sum_{k=1}^{N-1}(D+J)^{-1}_{i,k}\tilde{S}_k \ \|T(h)\| = max|T(h)| \text{ for } 1,2,\ldots,N-1 \quad (25)$$

since $(D+J)^{-1}_{i,k} \geq 0$ and $\sum_{k=1}^{N-1}(D+J)^{-1}_{i,k}\tilde{S}_k = 1$ for $i = 1,2,\ldots,N-1,$

then $$(D+J)^{-1}_{i,k} \leq \frac{1}{\tilde{S}_i} < \frac{1}{h^2[(R_1+R_2+R_4)D_1{}^* - 2(R_1+R_4)\omega C^2_{1^*}]} \text{ for } i = 1 \quad (26)$$

$$(D+J)^{-1}_{i,k} \leq \frac{1}{\tilde{S}_i} < \frac{1}{h^2[(R_1+R_2+R_4)D_1{}^* - 2(R_1+R_4)\omega C^2_{1^*}]} \text{ for } i = N-1. \quad (27)$$

$$(D+J)^{-1}_{i,k} \leq \frac{1}{\tilde{S}_i} < \frac{1}{2h^2(R_1+R_2+R_3+R_4)D_{1^*}} \text{ for } i = 2,3,\ldots,N-2 \quad (28)$$

with the help of the Eqs. (26) - (28) and using the Eq. (21), it is clear that

$$\|E\| \leq O(h^4)$$

hence the proposed method is fourth order convergent on uniform mesh for $R_2 = R_3 = \frac{1}{12}$, $R_1 + R_4 = \frac{5}{12}$ and $\omega = \frac{-1}{20\varepsilon}$

7. Numerical Examples

To test the viability of the proposed method based on adaptive spline and to demonstrate computationally their convergence, the following problems are chosen.

Example 7.1: $\varepsilon \frac{d^2w}{dv^2} + \frac{1}{v} w = g(v), \quad 0 < v < 1$, whose exact solution is $w(v) = v \sinh v$

Example 7.2: $\varepsilon \frac{d^2w}{dv^2} + \frac{1}{v}\frac{dw}{dv} + (1 + v^2)w = g(v), 0 < v < 1$, whose exact solution is $w(v) = e^{v^2}$

Table 1. The RMS error in Example 7.1 for various values of ε.

Present Method								
$h \downarrow$	$\varepsilon = 2^{-1}$	$\varepsilon = 2^{-3}$	$\varepsilon = 2^{-4}$	$\varepsilon = 2^{-5}$	$\varepsilon = 2^{-6}$	$\varepsilon = 2^{-8}$	$\varepsilon = 2^{-9}$	$\varepsilon = 2^{-10}$
2^{-3}	0.1461(-3)	0.8485(-3)	0.1266(-2)	0.5743(-2)	0.5788(-4)	0.5130(-3)	0.1810(-2)	0.8241(-3)
2^{-4}	0.1451(-3)	0.8290(-4)	0.1161(-3)	0.2756(-3)	0.1255(-3)	0.3063(-4)	0.3280(-5)	0.1062(-3)
2^{-5}	0.1362(-5)	0.7894(-5)	0.1091(-4)	0.2309(-4)	0.2654(-4)	0.2534(-4)	0.2428(-4)	0.2411(-4)
2^{-6}	0.1240(-6)	0.7320(-6)	0.1027(-5)	0.2058(-5)	0.3753(-5)	0.5955(-5)	0.3334(-5)	0.2206(-5)
2^{-7}	0.1115(-7)	0.6649(-7)	0.9524(-7)	0.1877(-6)	0.4543(-6)	0.1081(-5)	0.6673(-6)	0.9558(-6)
2^{-8}	0.9888(-9)	0.5961(-8)	0.8677(-8)	0.1710(-7)	0.4896(-7)	0.1548(-6)	0.1077(-6)	0.2708(-5)
2^{-9}	0.8767(-10)	0.5307(-9)	0.7802(-9)	0.1543(-8)	0.4812(-8)	0.2041(-7)	0.1541(-7)	0.2750(-7)
2^{-10}	0.7760(-11)	0.4707(-10)	0.695(-10)	0.1380(-9)	0.4473(-9)	0.2268(-8)	0.1914(-8)	0.1863(-8)
Results in [3]								
$h \downarrow$	$\varepsilon = 2^{-1}$	$\varepsilon = 2^{-3}$	$\varepsilon = 2^{-4}$	$\varepsilon = 2^{-5}$	$\varepsilon = 2^{-6}$	$\varepsilon = 2^{-8}$	$\varepsilon = 2^{-9}$	$\varepsilon = 2^{-10}$
2^{-3}	0.2143(-2)	0.2385(-2)	--	--	--	--	--	--
2^{-4}	0.5520(-3)	0.5035(-3)	0.1983(-2)	--	--	--	--	--
2^{-5}	0.1390(-3)	0.1212(-3)	0.3151(-3)	0.2485(-3)	0.2485(-3)	--	--	--
2^{-6}	0.3475(-4)	0.3010(-4)	0.6779(-4)	o.6301(-4)	0.6301(-4)	--	--	--
2^{-7}	0.8676(-5)	0.7512(-5)	0.1604(-4)	0.1949(-4)	0.1949(-4)	0.1642(-4)	--	--
2^{-8}	0.2166(-5)	0.1876(-5)	03932(-5)	0.5845(-5)	0.5845(-5)	0.5232(-5)	0.5434(-3)	--
2^{-9}	0.5411(-6)	0.4687(-6)	0.9765(-6)	0.1606(-5)	0.1606(-5)	0.2053(-5)	0.1419(-5)	0.5261(-5)
2^{-10}	0.1352(-6)	0.1171(-6)	0.2435(-6)	0.4170(-6)	0.4170(-6)	0.7589(-6)	0.8411(-6)	0.1281(-5)

Table 2. The RMS error in Example 7.2 for various values of ε.

$h \downarrow$	$\varepsilon = 2^{-2}$	$\varepsilon = 2^{-3}$	$\varepsilon = 2^{-4}$	$\varepsilon = 2^{-5}$	$\varepsilon = 2^{-6}$	$= 2^{-7}$	$\varepsilon = 2^{-8}$	$\varepsilon = 2^{-9}$	$\varepsilon = 2^{-10}$
2^{-4}	0.289(-3)	0.316(-3)	0.3340(-3)	0.3459(-3)	0.357(-3)	0.371(-3)	0.383(-3)	0.367(-3)	0.329(-3)

2^{-5}	0.272(-4)	0.295(-4)	0.3102(-4)	0.3184(-4)	0.324(-4)	0.329(-4)	0.337(-4)	0.348(-4)	0.361(-4)
2^{-6}	0.248(-5)	0.269(-5)	0.2814(-5)	0.2881(-5)	0.291(-5)	0.294(-5)	0.296(-5)	0.300(-5)	0.306(-5)
2^{-7}	0.272(-6)	0.241(-6)	0.2521(-6)	0.2579(-6)	0.260(-6)	0.262(-6)	0.263(-6)	0.264(-6)	0.266(-6)
2^{-8}	0.198(-7)	0.214(-7)	0.2244(-7)	0.2295(-7)	0.232(-7)	0.233(-7)	0.234(-7)	0.234(-7)	0.235(-7)
2^{-9}	0.175(-8)	0.190(-8)	0.1990(-8)	0.2035(-8)	0.205(-8)	0.207(-8)	0.207(-8)	0.207(-8)	0.208(-8)
2^{-10}	0.155(-9)	0.1687(-9)	0.1762(-9)	0.1802(-9)	0.182(-9)	0.183(-9)	0.183(-9)	0.184(-9)	0.184(-9)
Results in [3]									
2^{-4}	0.138(-1)	0.2297(-1)	0.3805(-1)	0.6476(-1)	--	--	--	--	--
2^{-5}	0.336(-2)	0.5742(-2)	0.9782(-2)	0.1720(-1)	0.311(-1)	0.568(-1)	--	--	--
2^{-6}	0.827(-3)	0.1433(-2)	0.2478(-2)	0.4427(-2)	0.816(-2)	0.153(-1)	0.290(-1)	0.546(-1)	--
2^{-7}	0.205(-3)	0.3582(-3)	0.6240(-3)	0.1123(-2)	0.209(-2)	0.399(-2)	0.769(-2)	0.148(-1)	0.284(-1)
2^{-8}	0.510(-4)	0.8953(-4)	0.1566(-3)	0.2834(-3)	0.530(-3)	0.101(-2)	0.198(-2)	0.387(-2)	0.756(-2)
2^{-9}	0.127(-4)	0.2238(-4)	0.3924(-4)	0.7119(-4)	0.133(-3)	0.257(-3)	0.503(-3)	0.990(-3)	0.195(-2)
2^{-10}	0.318(-5)	0.5594(-5)	0.9825(-5)	0.1784(-4)	0.336(-4)	0.648(-4)	0.127(-3)	0.250(-3)	0.496(-3)

8. Conclusion

In this paper, a fourth order accurate difference scheme has been discussed using an adaptive spline for the numerical solution of singularly perturbed two-point singular boundary-value problems. For non-singular problems also the proposed method is applicable. The convergence analysis of the method has been discussed. Numerical results are provided to demonstrate the efficiency of the proposed method. In the tables 1-2, the root mean square (RMS) errors in the solution are tabulated in comparison to the results given in [3]. From the numerical results, it observes that the proposed method works for smaller values of ε also.

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