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On Weighted Sums of Horadam Numbers

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Abstract:

For nonzero integers p, q with $p - q \neq 1$, let $(U_n)_{n\geq 0}$ be the Lucas sequence of the first kind defined by $U_n = pU_{n-1} - qU_{n-2}$ for $n \geq 2$ with initial terms $U_0 = 0$ and $U_1 = 1$. In this paper, we provide explicit formulas for evaluating the weighted sums $\sum_{k=1}^{n} k^m U_k$ and $\sum_{k=1}^{n} k^m U_{k+r}$ using binomial coefficients for nonnegative integers m and arbitrary integers r. Furthermore, we extend these summation formulas to the Horadam sequence $(W_n)_{n\geq 0}$ defined by the same recursion $W_n = pW_{n-1} - qW_{n-2}$ for $n \geq 2$ with arbitrary initial terms W_0 and W_1.

Keywords: Binomial coefficient, Brousseau sum, Fibonacci number, Horadam number, Lucas number.

1. Introduction

Let p and q be nonzero integers with $\lambda = p - q - 1 \neq 0$. The Lucas sequences [11], $(U_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$, are defined by

$$U_0 = 0, \ \ U_1 = 1, \ \ U_n = pU_{n-1} - qU_{n-2} \quad \text{for} \quad n \geq 2,$$

and

$$V_0 = 2, \ \ V_1 = p, \ \ V_n = pV_{n-1} - qV_{n-2} \quad \text{for} \quad n \geq 2,$$

respectively. These numbers can be extended to negative indices by

$$U_n = \frac{pU_{n+1} - U_{n+2}}{q} \quad \text{and} \quad V_n = \frac{pV_{n+1} - V_{n+2}}{q},$$

for $n < 0$. Some special cases of U_n and V_n are given in Table 1 and Table 2, respectively.

Table 1: Examples of Lucas sequences of the first kind (U_n)

(p, q)	Recurrence name	First few terms starting at $n = 0$	OEIS [14]
$(1, -1)$	Fibonacci	$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$	A000045
$(2, -1)$	Pell	$0, 1, 2, 5, 12, 29, 70, 169, 408, \ldots$	A000129
$(6, 1)$	Balancing	$0, 1, 6, 35, 204, 1189, 6930, 40391, \ldots$	A001109
$(1, -2)$	Jacobsthal	$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, \ldots$	A001045

Table 2: Examples of Lucas sequences of the second kind (V_n)

(p, q)	Recurrence name	First few terms starting at $n = 0$	OEIS [14]
$(1, -1)$	Lucas	$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$	A000032
$(2, -1)$	Pell-Lucas	$2, 2, 6, 14, 34, 82, 198, 478, 1154, \ldots$	A002203
$(6, 1)$	Balancing-Lucas	$2, 6, 34, 198, 1154, 6726, 39202, \ldots$	A003499
$(1, -2)$	Jacobsthal-Lucas	$2, 1, 5, 7, 17, 31, 65, 127, 257, 511, \ldots$	A014551

The Horadam sequence [8], $W_n = w_n(W_0, W_1; p, q)$, is a generalization of the Lucas sequences and is defined by

$$W_n = pW_{n-1} - qW_{n-2} \qquad \text{for} \ \ n \geq 2,$$

with initial terms W_0 and W_1. Clearly, $U_n = w_n(0,1;p,q)$ and $V_n = w_n(2,p;p,q)$. When $p = 1$ and $q = -1$, the Lucas sequences $(U_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$ are reduced respectively to the classical Fibonacci sequence $(F_n)_{n\geq 0}$ and the companion Lucas sequence $(L_n)_{n\geq 0}$. Many authors have attempted to find an exact formula for the sum

$$\sum_{k=1}^{n} k^m F_k,$$

using various techniques, where m is a nonnegative integer. These methods include linear recurrence relations, the finite difference approach, linear operators, and matrix methods. The study started in 1963 with a problem proposed by Brousseau [3] to discover an expression for the sum $\sum_{i=1}^{n} i^3 F_i$. One year later, Erbacher and Fuchs [6] proposed a solution for this problem. Later, Ledin [10], Brousseau [4], and Zeitlin [16], and recently, Ollerton and Shannon [12], Shannon and Ollerton [14], Dresden [5], and Adegoke [1] derived expressions for such sums. Dresden [5] used a new method in which only binomial coefficients were used to find the sum. Nair and Karunakaran [13] applied similar methods to extend Dresden's formula to k-Fibonacci numbers. This article further extends their work by deriving an explicit formula for the Lucas-Brousseau sum

$$\sum_{k=1}^{n} k^m U_k.$$

This is our Theorem 2, which states that

$$\lambda \sum_{k=1}^{n} k^m U_k = \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right) U_{n+1} - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right) U_n - \left(\mathcal{C}_{(p,q)}^{(m)}(0)\right), \qquad (1)$$

where $\mathcal{C}_{(p,q)}^{(m)}(n)$ is a *coefficient polynomial*, a polynomial in n of degree m. We find this polynomial using simple recursive formulas involving only binomial coefficients. Specifically, we show that

$$\mathcal{C}_{(p,q)}^{(m)}(n) = \sum_{j=0}^{m} \mathcal{A}_{(p,q)}^{(j)} \binom{m}{j} n^{m-j},$$

where the numbers $\mathcal{A}_{(p,q)}^{(m)}$ are defined by $\mathcal{A}_{(p,q)}^{(0)} = 1$ and

$$\mathcal{A}_{(p,q)}^{(m)} = \frac{1}{\lambda} \sum_{j=1}^{m} \left((-1)^j + q\right) \binom{m}{j} \mathcal{A}_{(p,q)}^{(m-j)}, \quad \text{for } m \geq 1.$$

Furthermore, we extend (1) to the Horadam sequence $(W_n)_{n\geq 0}$ and obtain (Theorem 5)

$$\lambda \sum_{k=1}^{n} k^m W_k = \mathcal{C}_{(p,q)}^{(m)}(n) W_{n+1} - q\mathcal{C}_{(p,q)}^{(m)}(n+1) W_n - \mathcal{C}_{(p,q)}^{(m)}(0) W_1 + q\mathcal{C}_{(p,q)}^{(m)}(1) W_0.$$

Throughout this paper, we assume that $\binom{0}{0} = 1$ and $0^0 = 1$.

2. Weighted sums in recurrence form

In this section, we derive a recurrence formula for the Brousseau sums

$$\sum_{k=1}^{n} k^m U_k$$

of the Lucas numbers of the first kind. Using this, along with the known identity

$$(p-q-1)\sum_{k=1}^{n} U_k = U_{n+1} - qU_n - 1, \qquad (2)$$

we can find the Brousseau sums of the Lucas numbers for $m = 1, 2, 3, \ldots$.

Theorem 1. *For all integers $m, n \geq 1$, the following identity holds:*

$$(p-q-1)\sum_{k=1}^{n} k^m U_k = n^m U_{n+1} - q(n+1)^m U_n + \sum_{j=1}^{m}\left((-1)^j + q\right)\binom{m}{j}\left(\sum_{k=1}^{n} k^{m-j} U_k\right).$$

Proof. Since $U_{k+1} = pU_k - qU_{k-1}$ for all $k \geq 1$, we have

$$pU_k = U_{k+1} + qU_{k-1}. \qquad (3)$$

Multiplying Eq. (3) through by k^m and then adding each side for $1 \leq k \leq n$, we obtain

$$\begin{aligned} p\sum_{k=1}^{n} k^m U_k &= \sum_{k=1}^{n} k^m U_{k+1} + q\sum_{k=1}^{n} k^m U_{k-1} \\ &= \sum_{k=2}^{n+1} (k-1)^m U_k + q\sum_{k=0}^{n-1} (k+1)^m U_k \\ &= \sum_{k=1}^{n+1} (k-1)^m U_k + q\sum_{k=1}^{n-1} (k+1)^m U_k. \end{aligned}$$

After modifying the summations on the right-hand side so that the variable k ranges from 1 to n, we obtain

$$p\sum_{k=1}^{n} k^m U_k = n^m U_{n+1} - q(n+1)^m U_n + \sum_{k=1}^{n}\left((k-1)^m + q(k+1)^m\right) U_k. \qquad (4)$$

By using the binomial expansion, we have

$$\begin{aligned} (k-1)^m + q(k+1)^m &= \sum_{j=0}^{m}\left((-1)^j + q\right)\binom{m}{j} k^{m-j} \\ &= (1+q)k^m + \sum_{j=1}^{m}\left((-1)^j + q\right)\binom{m}{j} k^{m-j}. \end{aligned}$$

Substituting this in Eq. (4),

we get

$$\begin{aligned} p\sum_{k=1}^{n} k^m U_k = n^m U_{n+1} &- q(n+1)^m U_n + (1+q)\sum_{k=1}^{n} k^m U_k \\ &+ \sum_{k=1}^{n}\sum_{j=1}^{m}\left((-1)^j + q\right)\binom{m}{j} k^{m-j} U_k. \end{aligned}$$

Thus,

$$(p-q-1)\sum_{k=1}^{n} k^m U_k = n^m U_{n+1} - q(n+1)^m U_n + \sum_{j=1}^{m}\left((-1)^j + q\right)\binom{m}{j}\left(\sum_{k=1}^{n} k^{m-j} U_k\right).$$

This completes the proof. □

Setting $m=1$ in Theorem 1 yields

$$(p-q-1)\sum_{k=1}^{n} k\,U_k = nU_{n+1} - q(n+1)U_n + (q-1)\sum_{k=1}^{n} U_k. \qquad (4)$$

Substituting Eq. (2) in Eq. (4), we get

$$(p-q-1)\sum_{k=1}^{n} k\,U_k = nU_{n+1} - q(n+1)U_n + (q-1)\frac{U_{n+1}-qU_n-1}{p-q-1}$$

$$= \left[n + \frac{q-1}{p-q-1}\right]U_{n+1} - q\left[n + \frac{p-2}{p-q-1}\right]U_n - \left[\frac{q-1}{p-q-1}\right]. \quad (5)$$

Likewise, by setting $m=2$ in Theorem 1 and using Eqs. (2), (4) and (5), we obtain

$$(p-q-1)\sum_{k=1}^{n} k^2 U_k = \left[n^2 + 2\left(\frac{q-1}{p-q-1}\right)n + \left(\frac{q^2+(p-6)q+(p+1)}{(p-q-1)^2}\right)\right]U_{n+1}$$

$$-q\left[n^2 + 2\left(\frac{p-2}{p-q-1}\right)n + \left(\frac{p^2+(q-3)p-4(q-1)}{(p-q-1)^2}\right)\right]U_n$$

$$-\left[\frac{q^2+(p-6)q+(p+1)}{(p-q-1)^2}\right]. \qquad (6)$$

3. Weighted sums in explicit form

In this section, we find an explicit formula for finding the Brousseau sums $\sum_{k=1}^{n} k^m U_k$ of the Lucas numbers of the first kind. We can rewrite Eqs. (2), (5), and (6) as

$$\lambda\sum_{k=1}^{n} U_k = U_{n+1} - qU_n - 1,$$

$$\lambda\sum_{k=1}^{n} k\,U_k = \left[n + \frac{q-1}{p-q-1}\right]U_{n+1} - q\left[(n+1) + \frac{q-1}{p-q-1}\right]U_n - \left[\frac{q-1}{p-q-1}\right],$$

$$\lambda\sum_{k=1}^{n} k^2 U_k = \left[n^2 + 2\left(\frac{q-1}{p-q-1}\right)n + \left(\frac{q^2+(p-6)q+(p+1)}{(p-q-1)^2}\right)\right]U_{n+1}$$

$$-q\left[(n+1)^2 + 2\left(\frac{q-1}{p-q-1}\right)(n+1) + \left(\frac{q^2+(p-6)q+(p+1)}{(p-q-1)^2}\right)\right]U_n$$

$$-\left[\frac{q^2+(p-6)q+(p+1)}{(p-q-1)^2}\right].$$

From the above equations, we recognize that each equation is of the form

$$\lambda \sum_{k=1}^{n} k^m U_k = \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right) U_{n+1} - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right) U_n - \left(\mathcal{C}_{(p,q)}^{(m)}(0)\right),$$

where $\mathcal{C}_{(p,q)}^{(m)}(n)$ is a polynomial in n of degree m. To understand the nature of the coefficients of this polynomial, we must define the following sequence $(A^{(m)})_{m\geq 0}$:

$$\mathcal{A}_{(p,q)}^{(m)} = \begin{cases} 1, & \text{if } m = 0\,; \\ \frac{1}{\lambda}\sum_{j=1}^{m}\left((-1)^j + q\right)\binom{m}{j}\mathcal{A}_{(p,q)}^{(m-j)}, & \text{if } m \geq 1\,. \end{cases}$$

Note that here we assumed that $\lambda = p - q - 1 \neq 0$.

The recurrence $\mathcal{A}_{(p,q)}^{(m)}$ generates the sequence

$$1, \frac{q-1}{\lambda}, \frac{q^2+(p-6)q+p+1}{\lambda^2}, \frac{q^3+(4p-23)q^2+(p^2+23)q-p^2-4p-1}{\lambda^3}, \ldots.$$

Table 3: Coefficient sequences of some classical sequences

(p,q)	Recurrence	Coefficient Sequence $\mathcal{A}_{(p,q)}^{(m)}$
$(1,-1)$	Fibonacci/Lucas	$1,\ -2,\ 8,\ -50,\ 416,\ -4322,\ 53888,\ \ldots$
$(1,-2)$	Pell/Pell-Lucas	$1,\ -1,\ 2,\ -7,\ 32,\ -181,\ 1232,\ -9787,\ \ldots$
$(2,-1)$	Jacobsthal/Jacobsthal-Lucas	$1,\ -\frac{3}{2},\ 4,\ -\frac{69}{4},\ 100,\ -\frac{1443}{2},\ \ldots$
$(6,1)$	Balancing numbers	$1,\ 0,\ \frac{1}{2},\ 0,\ 2,\ 0,\ \frac{77}{4},\ 0,\ 347,\ \ldots$

From the last equation in in the above set of weighted sums, we see that

$$\mathcal{C}_{(p,q)}^{(2)}(n) = n^2 + 2\left[\frac{q-1}{\lambda}\right]n + \left[\frac{q^2+(p-6)q+(p+1)}{\lambda^2}\right]$$
$$= \binom{2}{0}\mathcal{A}_{(p,q)}^{(0)}n^2 + \binom{2}{1}\mathcal{A}_{(p,q)}^{(1)}n + \binom{2}{2}\mathcal{A}_{(p,q)}^{(2)}.$$

With this in mind, we define the following coefficient polynomial $\mathcal{C}_{(p,q)}^{(m)}(n)$ in n of degree m:

$$\mathcal{C}_{(p,q)}^{(m)}(n) = \sum_{j=0}^{m}\binom{m}{j}\mathcal{A}_{(p,q)}^{(j)}n^{m-j}. \tag{7}$$

Note that $\mathcal{C}_{(p,q)}^{(m)}(0) = \mathcal{A}_{(p,q)}^{(m)}$.

Now we are ready to state and prove the key formula.

Theorem 2. Let $\lambda = p - q - 1 \neq 0$. *Then, for all integers* m, n *with* $m \geq 0$ *and* $n \geq 1$, *the following identity holds:*

$$\lambda \sum_{k=1}^{n} k^m U_k = \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right) U_{n+1} - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right) U_n - \left(\mathcal{C}_{(p,q)}^{(m)}(0)\right). \tag{8}$$

That is,

$$\lambda\sum_{k=1}^{n} k^m U_k = \left[\sum_{r=0}^{m}\binom{m}{r}\mathcal{A}_{(p,q)}^{(r)} n^{m-r}\right]U_{n+1} - q\left[\sum_{r=0}^{m}\binom{m}{r}\mathcal{A}_{(p,q)}^{(r)}(n+1)^{m-r}\right]U_n - \mathcal{A}_{(p,q)}^{(0)}.$$

Proof. We use induction on m. The case where $m = 0$ follows from Eq. (2). Now, set $m \geq 1$. Assume that Eq. (8) holds for all nonnegative integers less than m. By Theorem 1, we have

$$\lambda\sum_{k=1}^{n} k^m U_k = n^m U_{n+1} - q(n+1)^m + \sum_{j=1}^{m}\left((-1)^j + q\right)\binom{m}{j}\left[\sum_{k=1}^{n} k^{m-j} U_k\right]. \qquad (9)$$

Applying induction assumption, for $j = 1,2,\dots,m$, we have

$$\lambda\sum_{k=1}^{n} k^{m-j} U_k = \left(\mathcal{C}_{(p,q)}^{(m-j)}(n)\right)U_{n+1} - q\left(\mathcal{C}_{(p,q)}^{(m-j)}(n+1)\right)U_n - \left(\mathcal{C}_{(p,q)}^{(m-j)}(0)\right). \qquad (10)$$

Substituting Eq. (10) in Eq. (9) yields

$$\begin{aligned}\lambda\sum_{k=1}^{n} k^m U_k &= n^m U_{n+1} - q(n+1)^m + \left[\frac{1}{\lambda}\sum_{j=1}^{m}\left((-1)^j + q\right)\binom{m}{j}\left(\mathcal{C}_{(p,q)}^{(m-j)}(n)\right)\right]U_{n+1} \\ &\quad - \left[\frac{q}{\lambda}\sum_{j=1}^{m}\left((-1)^j + q\right)\binom{m}{j}\left(\mathcal{C}_{(p,q)}^{(m-j)}(n+1)\right)\right]U_n \\ &\quad - \frac{1}{\lambda}\sum_{j=1}^{m}\left((-1)^j + q\right)\binom{m}{j}\left(\mathcal{C}_{(p,q)}^{(m-j)}(0)\right). \qquad (11)\end{aligned}$$

For any non-negative integer x, write

$$\sum(x) = \frac{1}{\lambda}\sum_{j=1}^{m}\left((-1)^j + q\right)\binom{m}{j}\left(\mathcal{C}_{(p,q)}^{(m-j)}(x)\right). \qquad (12)$$

Using this notation, we can rewrite Eq. (11) as

$$\lambda\sum_{k=1}^{n} k^m U_k = n^m U_{n+1} - q(n+1)^m + \left(\sum(n)\right)U_{n+1} - q\left(\sum(n+1)\right)U_n - \sum(0). \qquad (13)$$

Now, using Eq. (7), we have

$$\mathcal{C}_{(p,q)}^{(m-j)}(x) = \sum_{r=0}^{m-j}\binom{m-j}{r}\mathcal{A}_{(p,q)}^{(r)} x^{m-j-r}.$$

Substituting this in Eq. (12) yields

$$\sum(x) = \frac{1}{\lambda}\sum_{j=1}^{m}\left((-1)^j + q\right)\binom{m}{j}\left[\sum_{r=0}^{m-j}\binom{m-j}{r}\mathcal{A}_{(p,q)}^{(r)} x^{m-j-r}\right].$$

Now, executing the change of variable $r' = r + j$ (keeping j fixed), this becomes

$$\sum(x) = \frac{1}{\lambda}\sum_{j=1}^{m}\left((-1)^j + q\right)\binom{m}{j}\left[\sum_{r'=j}^{m}\binom{m-j}{r'-j}\mathcal{A}_{(p,q)}^{(r'-j)}x^{m-r'}\right].$$

By switching the order of summation and using the binomial identity (see [2, Identity 134]) $\binom{m}{j}\binom{m-j}{r'-j} = \binom{m}{r'}\binom{r'}{j}$, we obtain

$$\sum(x) = \sum_{r'=1}^{m}\binom{m}{r'}\left[\frac{1}{\lambda}\sum_{j=1}^{r'}\left((-1)^j + q\right)\binom{r'}{j}\mathcal{A}_{(p,q)}^{(r'-j)}\right]x^{m-r'} = \sum_{r'=1}^{m}\binom{m}{r'}\mathcal{A}_{(p,q)}^{(r')}x^{m-r'}.$$

Thus, $\sum(x) = \mathcal{C}_{(p,q)}^{m}(x) - x^m$.

Therefore,

$$\begin{aligned}\sum(n) &= \mathcal{C}_{(p,q)}^{m}(n) - n^m,\\ \sum(n+1) &= \mathcal{C}_{(p,q)}^{m}(n+1) - (n+1)^m,\\ \sum(0) &= \mathcal{C}_{(p,q)}^{m}(0).\end{aligned}$$

Substituting this in Eq. (13) yields

$$\begin{aligned}\lambda\sum_{k=1}^{n}k^m U_k &= n^m U_{n+1} - q(n+1)^m U_n + \left[\mathcal{C}_{(p,q)}^{m}(n) - n^m\right]U_{n+1}\\ &\quad - q\left[\mathcal{C}_{(p,q)}^{m}(n+1) - (n+1)^m\right]U_n - \mathcal{C}_{(p,q)}^{m}(0)\\ &= \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right)U_{n+1} - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right)U_n - \left(\mathcal{C}_{(p,q)}^{(m)}(0)\right).\end{aligned}$$

This completes the proof. □

Example 3. *As an illustration, we find the Brousseau sums $\sum_{k=1}^{n} k^4 B_k$ of the Balancing numbers. Setting $m = 4, p = 6, q = 1$, and $W_k = B_k$ in Eq. (8) yields*

$$4\cdot\sum_{k=0}^{n}k^4 B_k = \left(\mathcal{C}_{(6,1)}^{(4)}(n)\right)B_{n+1} - \left(\mathcal{C}_{(6,1)}^{(4)}(n+1)\right)B_n - \left(\mathcal{C}_{(6,1)}^{(4)}(0)\right),$$

where

$$\mathcal{C}_{(6,1)}^{(4)}(n) = \mathcal{A}_{(6,1)}^{(0)}\binom{4}{0}n^4 + \mathcal{A}_{(6,1)}^{(1)}\binom{4}{1}n^3 + \mathcal{A}_{(6,1)}^{(2)}\binom{4}{2}n^2 + \mathcal{A}_{(6,1)}^{(3)}\binom{4}{3}n^1 + \mathcal{A}_{(6,1)}^{(4)}\binom{4}{4}n^0.$$

Now, using Table 3, we obtain

$$\mathcal{C}_{(6,1)}^{(m)}(n) = n^4 + 3n^2 + 2.$$

Therefore,

$$\sum_{k=1}^{n}k^4 B_k = \frac{1}{4}\left[(n^4 + 3n^2 + 2)B_{n+1} - ((n+1)^4 + 3(n+1)^2 + 2)B_n - 2\right].$$

Theorem 4. Let $\lambda = p - q - 1 \neq 0$. *For all integers* m, n, r *with* $m \geq 0$ *and* $n \geq 1$, *we have*

$$\lambda \sum_{k=1}^{n} k^m U_{k+r} = \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right) U_{n+r+1} - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right) U_{n+r} - \left(\mathcal{C}_{(p,q)}^{(m)}(0)\right) U_{r+1} + q\left(\mathcal{C}_{(p,q)}^{(m)}(1)\right) U_r. \quad (14)$$

Proof. The case $r = 0$ follows from Theorem 2. Now, assume that $r \geq 1$. Using binomial expansion, we have

$$\begin{aligned}\sum_{k=1}^{n} k^m U_{k+r} &= \sum_{k=r+1}^{n+r} (k-r)^m U_k \\ &= \sum_{k=1}^{n+r} (k-r)^m U_k - \sum_{k=1}^{r} (k-r)^m U_k \\ &= \sum_{k=1}^{n+r} \sum_{j=0}^{m} \binom{m}{j} k^{m-j} (-r)^j U_k - \sum_{k=1}^{r} \sum_{j=0}^{m} \binom{m}{j} k^{m-j} (-r)^j U_k.\end{aligned}$$

Thus,

$$\sum_{k=1}^{n} k^m U_{k+r} = \sum_{j=0}^{m} \binom{m}{j} (-r)^j \left[\sum_{k=1}^{n+r} k^{m-j} U_k - \sum_{k=1}^{r} k^{m-j} U_k\right]. \quad (15)$$

Now, using Theorem 2, we have

$$\lambda \sum_{k=1}^{n+r} k^{m-j} U_k = \left(\mathcal{C}_{(p,q)}^{(m-j)}(n+r)\right) U_{n+r+1} - q\left(\mathcal{C}_{(p,q)}^{(m-j)}(n+r+1)\right) U_{n+r} - \mathcal{C}_{(p,q)}^{(m-j)}(0),$$

and

$$\lambda \sum_{k=1}^{r} k^{m-j} U_k \quad = \left(\mathcal{C}_{(p,q)}^{(m-j)}(r)\right) U_{n+r+1} - q\left(\mathcal{C}_{(p,q)}^{(m-j)}(r+1)\right) U_{n+r} - \mathcal{C}_{(p,q)}^{(m-j)}(0).$$

Substituting in (15), we obtain

$$\begin{aligned}&\lambda \sum_{k=1}^{n} k^m U_{k+r} \\ &= \left(\sum_{j=0}^{m} \binom{m}{j} (-r)^j \mathcal{C}_{(p,q)}^{(m-j)}(n+r)\right) U_{n+r+1} - q\left(\sum_{j=0}^{m} \binom{m}{j} (-r)^j \mathcal{C}_{(p,q)}^{(m-j)}(n+r+1)\right) U_{n+r} \\ &- \left(\sum_{j=0}^{m} \binom{m}{j} (-r)^j \mathcal{C}_{(p,q)}^{(m-j)}(r)\right) U_{r+1} + q\left(\sum_{j=0}^{m} \binom{m}{j} (-r)^j \mathcal{C}_{(p,q)}^{(m-j)}(r+1)\right) U_r. \quad (16)\end{aligned}$$

Now, for $x \in \mathbb{Z}$, let

$$\mathcal{S}(x) = \sum_{j=0}^{m} \binom{m}{j} (-r)^j \mathcal{C}_{(p,q)}^{(m-j)}(x). \tag{17}$$

Using this notation, we can rewrite (16) as

$$\lambda \sum_{k=1}^{n} k^m U_{k+r} = \mathcal{S}(n+r)U_{n+r+1} - q\mathcal{S}(n+r+1)U_{n+r} - \mathcal{S}(r)U_{r+1} + q\mathcal{S}(r+1)U_r. \tag{18}$$

Using Eqs. (7) and (17), we obtain

$$\mathcal{S}(x) = \sum_{j=0}^{m} \sum_{i=0}^{m-j} \binom{m}{j} \binom{m-j}{i} (-r)^j \mathcal{A}_{(p,q)}^{(i)} x^{m-j-i}$$

By switching the order of summation, this becomes

$$\mathcal{S}(x) = \sum_{i=0}^{m} \sum_{j=0}^{m-i} \binom{m}{j} \binom{m-j}{i} (-r)^j \mathcal{A}_{(p,q)}^{(i)} x^{m-j-i}$$

Now, using the binomial identity (see [2, Identity 134]) $\binom{m}{j}\binom{m-j}{i} = \binom{m}{i}\binom{m-i}{j}$, we obtain

$$\mathcal{S}(x) = \sum_{i=0}^{m} \binom{m}{j} \mathcal{A}_{(p,q)}^{(i)} \left[\sum_{j=0}^{m-i} \binom{m-j}{i} (-r)^j x^{m-j-i} \right]$$

$$= \sum_{i=0}^{m} \binom{m}{j} \mathcal{A}_{(p,q)}^{(i)} (x-r)^{m-i},$$

where the last equality follows from the binomial expansion. Thus,

$$\mathcal{S}(x) = \mathcal{C}^{(m)}(x-r).$$

Therefore,

$$\begin{aligned} \mathcal{S}(n+r) &= \mathcal{C}_{(p,q)}^{(m)}(n), \\ \mathcal{S}(n+r+1) &= \mathcal{C}_{(p,q)}^{(m)}(n+1), \\ \mathcal{S}(r) &= \mathcal{C}_{(p,q)}^{(m)}(0), \\ \mathcal{S}(r+1) &= \mathcal{C}_{(p,q)}^{(m)}(1). \end{aligned}$$

Substituting in Eq. (18), we obtain

$$\lambda \sum_{k=1}^{n} k^m U_{k+r} = \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right) U_{n+r+1} - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right) U_{n+r}$$
$$- \left(\mathcal{C}_{(p,q)}^{(m)}(0)\right) U_{r+1} + q\left(\mathcal{C}_{(p,q)}^{(m)}(1)\right) U_r,$$

as desired.

Next, we consider the case $r \leq -1$. Now, using the identity [9, Identity 4.19], we have

$$U_{k+r} = U_{k+1}U_r - qU_kU_{r-1}. \tag{19}$$

Multiplying Eq. (19) through by k^m, and then adding each side for $1 \leq k \leq n$, we obtain

$$\sum_{k=1}^{n} k^m U_{k+r} = U_r \sum_{k=1}^{n} k^m U_{k+1} - qU_{r-1} \sum_{k=1}^{n} k^m U_k. \tag{20}$$

Now, applying the above part of the proof, we obtain

$$\lambda \sum_{k=1}^{n} k^m U_k = \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right) U_{n+1} - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right) U_n - \left(\mathcal{C}_{(p,q)}^{(m)}(0)\right), \tag{21}$$

and

$$\lambda \sum_{k=1}^{n} k^m U_{k+1} = \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right) U_{n+2} - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right) U_{n+1} - p\left(\mathcal{C}_{(p,q)}^{(m)}(0)\right) + q\left(\mathcal{C}_{(p,q)}^{(m)}(1)\right). \tag{22}$$

Using Eqs. (20) - (22), we obtain

$$\begin{aligned}
\lambda \sum_{k=1}^{n} k^m U_{k+r} &= \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right)(U_rU_{n+2} - qU_{r-1}U_{n+1}) \\
&\quad - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right)(U_rU_{n+1} - qU_{r-1}U_n) \\
&\quad - \left(\mathcal{C}_{(p,q)}^{(m)}(0)\right)(pU_r - qU_{r-1}) + qU_r\left(\mathcal{C}_{(p,q)}^{(m)}(1)\right) \\
&= \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right) U_{n+r+1} - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right) U_{n+r} \\
&\quad - \left(\mathcal{C}_{(p,q)}^{(m)}(0)\right) U_{r+1} + q\left(\mathcal{C}_{(p,q)}^{(m)}(1)\right) U_r,
\end{aligned}$$

where the last equality follows from the identity (19) and the Lucas recurrence. This completes the proof. □

4. Extension to Horadam sequence

The Lucas numbers U_n and the Horadam numbers W_n are connected by the relation [9, Identity 2.14]

$$W_n = U_nW_1 - qU_{n-1}W_0. \tag{23}$$

Using identity (23), we can extend Theorem 4 to Horadam numbers.

Theorem 5. *Let $\lambda = p - q - 1 \neq 0$. Then for all integers m, n with $m \geq 0$ and $n \geq 1$, we have*

$$\lambda \sum_{k=1}^{n} k^m W_k = \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right) W_{n+1} - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right) W_n - \left(\mathcal{C}_{(p,q)}^{(m)}(0)\right) W_1 + q\left(\mathcal{C}_{(p,q)}^{(m)}(1)\right) W_0. \tag{24}$$

Proof. Since $W_k = U_k W_1 - qU_{k-1}W_0$, we have

$$\sum_{k=1}^{n} k^m W_k = W_1 \sum_{k=1}^{n} k^m U_k - qW_0 \sum_{k=1}^{n} k^m U_{k-1} \tag{25}$$

Now, using Theorem 4, we have

$$\lambda \sum_{k=1}^{n} k^m U_k = \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right) U_{n+1} - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right) U_n - \left(\mathcal{C}_{(p,q)}^{(m)}(0)\right)$$

and $$\lambda \sum_{k=1}^{n} k^m U_{k-1} = \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right) U_n - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right) U_{n-1} - \left(\mathcal{C}_{(p,q)}^{(m)}(1)\right).$$

Substituting in Eq. (25), we obtain

$$\begin{aligned}\lambda \sum_{k=1}^{n} k^m W_k &= \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right)(U_{n+1}W_1 - qU_nW_0) - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right)(U_nW_1 - qU_{n-1}W_0) \\ &\quad - \left(\mathcal{C}_{(p,q)}^{(m)}(0)\right)W_1 + q\left(\mathcal{C}_{(p,q)}^{(m)}(1)\right)W_0 \\ &= \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right)W_{n+1} - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right)W_n - \left(\mathcal{C}_{(p,q)}^{(m)}(0)\right)W_1 + q\left(\mathcal{C}_{(p,q)}^{(m)}(1)\right)W_0,\end{aligned}$$

where the last equality follows from Eq. (23). This completes the proof. □

Setting $W_k = V_k$ in Theorem 5, we have the following corollary:

Corollary 6. *Let $\lambda = p - q - 1 \neq 0$. Then for all integers $m \geq 0$, we have*

$$\lambda \sum_{k=1}^{n} k^m V_k = \left(\mathcal{C}_{(p,q)}^{(m)}(n)\right) V_{n+1} - q\left(\mathcal{C}_{(p,q)}^{(m)}(n+1)\right) V_n - p\left(\mathcal{C}_{(p,q)}^{(m)}(0)\right) + 2q\left(\mathcal{C}_{(p,q)}^{(m)}(1)\right).$$

Example 7. *As an illustration, we find the Brousseau sums $\sum_{k=1}^{n} k^3 L_k$ of the "regular" Lucas numbers L_k defined by $L_k = L_{k-1} + L_{k-2}, L_0 = 2, L_1 = 1$. Setting $m = 3, p = 1, q = -1$, and $W_k = L_k$ in Eq. (23) yields*

$$\sum_{k=1}^{n} k^3 L_k = \left(\mathcal{C}_{(1,-1)}^{(3)}(n)\right) l_{n+1} + \left(\mathcal{C}_{(1,-1)}^{(3)}(n+1)\right) L_n - \left(\mathcal{C}_{(1,-1)}^{(3)}(0)\right) L_1 - \left(\mathcal{C}_{(1,-1)}^{(3)}(1)\right) L_0,$$

where

$$\mathcal{C}_{(1,-1)}^{(3)}(n) = \mathcal{A}_{(1,-1)}^{(0)}\binom{3}{0}n^3 + \mathcal{A}_{(1,-1)}^{(1)}\binom{3}{1}n^2 + \mathcal{A}_{(1,-1)}^{(2)}\binom{3}{2}n^1 + \mathcal{A}_{(1,-1)}^{(3)}\binom{3}{3}n^0.$$

Now, using Table 3, we obtain

$$\mathcal{C}_{(1,-1)}^{(3)}(n) = n^3 - 6n^2 + 24n - 50.$$

Therefore,

$$\sum_{k=1}^{n} k^3 L_k = (n^3 - 6n^2 + 24n - 50)L_{n+1} + ((n+1)^3 - 6(n+1)^2 + 24(n+1) - 50)L_n + 112.$$

5. Conclusion

For all integers m, n, r with $m \geq 0$ and $n \geq 1$, we have established an explicit formula for the sums $\sum_{k=1}^{n} k^m U_k$ and $\sum_{k=1}^{n} k^m U_{k+r}$, using just elementary methods. Furthermore, the study was extended to the Horadam numbers W_n. Both Ledin and Adegoke extended the study to the Lucas numbers. Our approach has the advantage of not requiring the complicated integration techniques of Ledin or the derivative methods of Adegoke. To obtain the polynomial form of the sum $\sum_{k=1}^{n} k^m W_k$, Adegoke employed two complicated sets of polynomials, $\mathcal{P}_1(m, n; p, q)$ and $\mathcal{P}_2(m, n; p, q)$, along with another complicated set of numbers, $\mathcal{C}(m, r; a, b, p, q)$. Our insight is that we can find these formulas using a single set of "coefficient polynomials" $\mathcal{C}_{p,q}^{(m)}(n)$.

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