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Numerical Simulation of Nonlinear Equations by using Modified Newton Raphson Technique

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Abstract:

Non-linear equations are a significant number of complicated issues in mathematics and related subjects that must be solved. The Newton method and its preliminary variations are among the most basic, yet insufficiently effective, methods for resolving non-linear equations. To solve non-linear equations efficiently, a method's desirable feature is to find root with fewer iterations, minimum error (usually less than precision limit), which is typically less than the precision limit. In, this work, we present two approaches (Proposed techniques I and II) for solving non-linear equations by modifying Newton Raphson's technique using first derivative's forward and central difference approximations. Two examples are used to compare performance of the suggested techniques with the existing technique (Secant technique). Finding roots of nonlinear equations under consideration required fewer iterations and a lower absolute relative approximate error while using proposed method II, which performed better than both existing methods (the Secant technique and suggested method I). The suggested approaches I and II were determined to be appropriate substitutes for solving nonlinear equations.

Keywords: Secant technique, forward difference approximation, Central difference approximation, nonlinear equation, numerical examples.

1. Introduction

The majority of the things in everyday life are represented by non-linear equations. Maheshwari (2009) asserts that functions of a nonlinear and transcendental nature can be found in many difficult situations in science and engineering when they are expressed as equations of the form $f(x) = 0$. [11]. As a result, their answers are crucial for addressing a wide range of issues that face us. Nonlinear equations have a large number of variables and their related parameters, which can lead to complex problems that are mostly solved numerically. Therefore, over the years, many numerical techniques have been developed that assist in the effective solution of these equations. The Bisection, Newton Raphson, and Secant Methods are some of these techniques. It is remarkable that the most common way for creating multipoint methods is Newton's method. (Petkovic, 2012) [12].

Kaw (2009) states that the bisection approach, also known as the binary-search method, was among the earliest numerical techniques created to determine the root of a nonlinear equation [9]. The technique is based on a theorem that says, "If $f(x_m)\, f(x_n) < 0$, then an equation $f(x) = 0$, where $f(x)$ is a real valued continuous function, has at least one root between x_m and x_n." where the first two initial approximations of the root of nonlinear equation $f(x) = 0$ are x_m and x_n.

Kaw (2009) enumerated the following benefits of this method: the method is guaranteed to converge because it brackets the roots. That is, since they depend on minimising the difference between the two estimates in order to find the equations' roots, they are always convergent [9].

As iterations are carried out, the interval is also halved. Therefore, the error in the equation's solution can be guaranteed.

The Bisection approach has some limitations, as noted by Chhabra (2014) and Kaw (2009) [4,9];

1. The Bisection technique delayed convergence can be attributed to its basic halving of the interval.

2. The method's computational efficiency decreases if a closer approximation to the root requires more iterations to find the root.

3. The Bisection technique may converge to a singularity for any function $f(x)$ in the presence of singularity.

The Newton Raphson technique was developed to solve nonlinear equations, addressing the drawbacks of the previously described Bisection approach.

A method that is open is the Newton Raphson method. This shows that the iterative process of finding roots of the equations may be started with just one initial guess.

In a tract, Newton described his method for approximating the fundamental causes of numerical equations. In 1600, Viet Nam created the perturbation technique to solve a scalar polynomial that provided one decimal place of an unknown answer for each step by explicit calculation of subsequent polynomials of sequential perturbations. Modernized terminology would say that the approach converged linearly. The Persian astronomer and mathematician Al-Kashi appears to have also published this technique in 1427. Al-Biruni's (1973–1048) considerably older work served as the foundation for The Key to Arithmetic; it is unclear how well known it was in Europe. In 1647, Oughtred, an English mathematician, simplified Vieta's approach. The Vieta technique was introduced to Isaac Newton (1643–1727) in 1664. He had enhanced it up until 1669 by linearizing the successively arising polynomials. De Analysi per Equations Numerariae Terminaraun Infinitas, a book written by Newton, explains his approach to approximating the fundamental causes of numerical equations. This is how the binomial theorem and the fluxions principle were first announced. Newton gave it to his tutor, Isaac Barrow, in 1669, who then gave it to J. Collins, who had a strong ambition to gather and disseminate experimental knowledge. The Royal Society included John Collin. The Wallis Algebra, published in London in 1685, chapter 94, represents the oldest attempt at using Newton's method of approximation. Wallis describes Newton's approach to resolving the equation. Newton's friends and some of Collins's correspondents were aware of the tract, although it wasn't printed until 1704 and 1711. In his second section, "The Methodus fluxionum et serierum infinitarum," Newton essentially explained his approximation approach in the same way. This was intended for publication in 1671, but it wasn't printed until 1736.

Finding the zero of $f(x) = 0$ is known as the "root finding problem," f being a single-variable function in this case. We want to find an $x = \omega$ such that $f(\omega) = 0$ for some function f. The root, or zero, of f is the number. The function f can be algebraic, transcendental, or exponential. One of the most important computational issues is the root determination issue. It appears in a range of real-world applications in the sciences, including physics, chemistry, biology, engineering, etc. Actually, determining any implicit unknown in mathematical or scientific formulas creates a problem known as root discovery [3]. Finding an object's equilibrium position, a field's potential surface, and the quantized energy level of a restricted structure are a few pertinent examples of circumstances in physics when solving such problems is necessary [6]. Bisection, Newton-Raphson, and other techniques are frequently used to discover roots. Various techniques converge on the root at various rates. In other words, some methodologies converge on the root more quickly than others. Convergence could occur at a linear, quadratic, or other rate. The faster the algorithm converges, the higher the order [2].

Material and Methods

Derivation of Newton Raphson technique from Taylor Series:

Cirnu (2012) states that the Taylor expansion's first order yields the Newton Raphson technique.

Assuming that initial guess for the root of $f(x) = 0$ is at x_k, an improved approximation of the root of the function $f(x_k)$, is x_{k+1} when tangent of the curve is drawn at $f(x_k)$, the point at which tangent line crosses x-axis.

For a nonlinear equation $f(x) = 0$, let, x_k, $k = 1,2,3 \ldots$ be an initial guess for the root. Then, define $x_{k+1} = x_k + \delta x$, where δx is a small change in the solution. Now the Taylor series expansion of a function $f(x)$ is given by

$$f(x_{k+1}) = \sum_{k=0}^{\infty} f^k(x_k)\left(\frac{x_{k+1}-x_k{}^k}{k!}\right) \ldots (1)$$

Where $f^0(x_k) = f(x_k)$, $f^1(x_k) = f'(x_k)$ and $f^2(x_k) = f''(x_k)$ and so on.

Where $f'(x_k)$ and $f''(x_k)$ represent the first and second order derivatives with respect to x of the function f respectively.

$$f(x_{k+1}) = f(x_k) + f'(x_k)\frac{(x_{k+1}-x_k)}{1!} + f''(x_k)\frac{(x_{k+1}-x_k)^2}{2!} + \ldots + \ldots (2)$$

$$f(x_{k+1}) = f(x_k) + f'(x_k)\frac{(x_{k+1}-x_k)}{1!} + o(\delta x)$$

where $o(\delta x)$ represents the error caused on by the Taylor series' truncation at the second term.

$$f(x_{k+1}) \approx f(x_k) + f'(x_k)\frac{(x_{k+1}-x_k)}{1!} \ldots (3)$$

Assume that, x_{k+1} is root of the equation, then $f(x_{k+1}) = 0$

After solving (3), we obtained

$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$ . . . (4) which is Newton Raphson formula.

First derivative Forward difference approximation:

First derivative forward difference approximation of the function f first can be obtained from equation (3) as follows:

$$f'(x_k) \approx \frac{f(x_{k+1})-f(x_k)}{x_{k+1}-x_k}$$

$$f'(x_k) \approx \frac{f(x_k+\delta x)-f(x_k)}{\delta x} \ldots (5)$$

First derivative Backward difference approximation:

For each $k = 1,2,\ldots$ Let $x_{k-1} = x_k - \delta x$, where δx represent small change in the solution, then Taylor series expansion of a function is given as

$$f(x_{k-1}) = f(x_k) - f'(x_k)\frac{(x_k-x_{k-1})}{1!} + f''(x_k)\frac{(x_k-x_{k-1})^2}{2!} + \ldots (6)$$

$$f(x_{k-1}) = f(x_k) - f'(x_k)\frac{(x_k-x_{k-1})}{1!} + o(\delta x)$$

$$f(x_{k-1}) \approx f(x_k) - f'(x_k)\frac{(x_k-x_{k-1})}{1!} \ldots (7)$$

Equation (7) yields first derivative backward difference approximation of the function f, which is given by

$$f'(x_k) \approx \frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}} \ldots (8)$$

$$f'(x_k) \approx \frac{f(x_k)-f(x_k-\delta x)}{\delta x} \ldots (9)$$

First derivative Central difference approximation:

Since $\delta x = x_{k-1} - x_k$ we can rewrite equation (2) as

$$f(x_{k+1}) = f(x_k) + f'(x_k)\frac{\delta x}{1!} + f''(x_k)\frac{(\delta x)^2}{2!} + \ldots + \ldots (10)$$

and also $\delta x = x_k - x_{k-1}$ then equation (6) can be written as

$$f(x_{k-1}) = f(x_k) - f'(x_k)\frac{(\delta x)}{1!} + f''(x_k)\frac{(\delta x)^2}{2!} + \ldots (11)$$

Subtracting equation (11) from equation (10), we get

$$f(x_{k+1}) - f(x_{k-1}) = 2f'(x_k)\,\delta x + 2f'''(x_k)\frac{(\delta x)^3}{3!} + \ldots$$

$$f(x_{k+1}) - f(x_{k-1}) = 2f'(x_k)\,\delta x + o(\delta x)^2$$

Where $o(\delta x)^2$ represents the error caused on by the Taylor series' truncation at third term.

$$f(x_{k+1}) - f(x_{k-1}) \approx 2f'(x_k)\,\delta x \ldots (12)$$

Equation (12) gives the first derivative central difference approximation of the function f, which is

$f'(x_k) \approx \frac{f(x_{k+1}) - f(x_{k-1})}{2\delta x}$

$f'(x_k) \approx \frac{f(x_k+\delta x) - f(x_k+\delta x)}{2\delta x}$ . . . (13)

Secant technique of solving nonlinear equations:

Substituting equation (8) in equation (4), we get

$x_{k+1} = x_k - f(x_k)[\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}]$

This is Secant technique for solving nonlinear equations.

Proposed Methodology I: Substituting equation (5) in equation (4), we get

$x_{k+1} = x_k - f(x_k)[\frac{\delta x}{f(x_k+\delta x) - f(x_k)}]$

$x_{k+1} = x_k - f(x_k)[\frac{x_k - x_{k-1}}{f(2x_k - x_{k-1}) - f(x_k)}]$, . . . (15) where $\delta x = x_k - x_{k-1}$

Proposed Methodology II: Substituting equation (13) in equation (4), we get

$x_{k+1} = x_k - f(x_k)[\frac{2\delta x}{f(x_k+\delta x) - f(x_k - \delta x)}]$

$x_{k+1} = x_k - f(x_k)[\frac{2(x_k - x_{k-1})}{f(x_k - x_{k-1}) - f(x_{k-1})}]$, . . . (16) where $\delta x = x_k - x_{k-1}$

Numerical Examples:

The primary performance metrics in this study implemented to access the numerical approaches are the absolute relative approximate error $|\epsilon_a|$, the number of significant digits that constitute the solution is m, and the number of numerical iterations required is n.

Starting with an initial guess x_0, stopping criterion may be given by

$|\epsilon_a| \leq 0.5 \times 10^{2-m}$

$m \leq 2 - \log_{10} 2|\epsilon_a|$

Illustration 1: Let $f(x) = \cos x - xe^x$

Solve above equation by Secant technique, Proposed methodology I and II by setting the initial approximations $x_0 = 1$ and $x_1 = 2$. Take the maximum number of iterations 10 and error tolerance is 10^{-3}.

Table 1: Solution from Secant technique

Iterations	Root	$\|\epsilon_a\|$	$f(x)$	m
1	314665×10^{-6}	217797952×10^{-6}	519871174×10^{-9}	-5
2	446728×10^{-6}	29562231×10^{-6}	203544778×10^{-9}	-3
3	531706×10^{-6}	15982091×10^{-6}	-42931093×10^{-9}	-2
4	516904×10^{-6}	2863468×10^{-6}	2592763×10^{-9}	0
5	517747×10^{-6}	162820×10^{-6}	30112×10^{-9}	3
6	517757×10^{-6}	1913×10^{-6}	-22×10^{-9}	7

7	517757×10^{-6}	1×10^{-6}	0	14

Table 2: Solution from Proposed methodology I

Iterations	Root	$\|\epsilon_a\|$	$f(x)$	m
1	832673×10^{-6}	2009521×10^{-5}	$-1241793162\times 10^{-9}$	-2
2	549698×10^{-6}	5147814×10^{-5}	-99796443×10^{-9}	-3
3	510267×10^{-6}	7727561×10^{-6}	22643969×10^{-9}	-1
4	518060×10^{-6}	1504182×10^{-6}	-919664×10^{-9}	0
5	517759×10^{-6}	57979×10^{-6}	-6207×10^{-9}	4
6	517757×10^{-6}	394×10^{-6}	2×10^{-9}	9

Table 3: Solution from Proposed methodology II

Iterations	Root	$\|\epsilon_a\|$	$f(x)$	m
1	731018×10^{-6}	36795478×10^{-6}	-773972497×10^{-9}	-3
2	553063×10^{-6}	32176402×10^{-6}	-110618158×10^{-9}	-3
3	519082×10^{-6}	6546315×10^{-6}	-4033931×10^{-9}	-1
4	517759×10^{-6}	255454×10^{-6}	-5839×10^{-9}	2
5	517757×10^{-6}	371×10^{-6}	0	9

From table 1, 2, and 3 we observed that the secant technique, proposed methodology I, II gives root in 7, 6 and 5 iterations respectively. This was achieved at an absolute relative approximate error, $|\epsilon_a|$ less than precision limit 10^{-3}.

Illustration 2: Let $f(x) = x^3 - e^{-x}$

Solve above equation by Secant technique, Proposed methodology I and II by setting initial approximations $x_0 = 0$ and $x_1 = 2$. Take the maximum number of iterations 10 and error tolerance is 10^{-3}.

Table 4: Solution from Secant technique

Iterations	Root	$\|\epsilon_a\|$	$f(x)$	m
1	225615×10^{-6}	786466472×10^{-6}	-786541141×10^{-9}	-6
2	386937×10^{-6}	41692024×10^{-6}	-621202127×10^{-9}	-3
3	993045×10^{-6}	61035345×10^{-6}	608832428×10^{-9}	-3
4	693038×10^{-6}	43288595×10^{-6}	-16718665×10^{-8}	-3
5	757672×10^{-6}	8530581×10^{-6}	-33801676×10^{-9}	-1
6	774051×10^{-6}	2116029×10^{-6}	2635996×10^{-9}	0
7	772866×10^{-6}	153314×10^{-6}	-37316×10^{-9}	3
8	772883×10^{-6}	214×10^{-5}	-4×10^{-8}	7
9	772883×10^{-6}	2×10^{-6}	0	14

Table 5: Solution from Proposed methodology I

Iterations	Root	$\|\epsilon_a\|$	$f(x)$	m
1	1719705×10^{-6}	16299028×10^{-6}	4906709767×10^{-9}	-2
2	1083411×10^{-6}	58730648×10^{-6}	933245222×10^{-9}	-3

3	683083×10^{-6}	58606065×10^{-6}	-186329777×10^{-9}	-3
4	820009×10^{-6}	16698124×10^{-6}	110958135×10^{-9}	-2
5	78016×10^{-5}	5107848×10^{-6}	16510283×10^{-9}	-1
6	772661×10^{-6}	970458×10^{-6}	-499788×10^{-9}	1
7	772885×10^{-6}	28904×10^{-6}	3587×10^{-9}	4
8	772883×10^{-6}	206×10^{-6}	1×10^{-9}	9

Table 6: Solution from Proposed methodology II

Iterations	Root	$\lvert\epsilon_a\rvert$	$f(x)$	m
1	1515884×10^{-6}	31936198×10^{-6}	3263743963×10^{-9}	-3
2	1072222×10^{-6}	41377791×10^{-6}	890444758×10^{-9}	-3
3	849578×10^{-6}	26206468×10^{-6}	185615279×10^{-9}	-2
4	77943×10^{-5}	89999×10^{-4}	1484507×10^{-8}	-1
5	772938×10^{-6}	839976×10^{-6}	122994×10^{-9}	1
6	772883×10^{-6}	706×10^{-5}	9×10^{-9}	6
7	772883×10^{-6}	0	0	15

From table 4, 5, and 6 we observed that secant technique, proposed methodology I, II gives the root in 9, 8 and 7 iterations respectively. This was achieved at an absolute relative approximate error, $\lvert\epsilon_a\rvert$ less than precision limit 10^{-3}.

Result discussion and conclusions

The study proposed a variation of the Newton Raphson technique using forward and central difference approximations of the first derivative. Proposed Method II outperformed Proposed Technique I and Secant method. It was developed from the improved Newton Raphson method of solving nonlinear equations using the first derivative's central difference approximation.

Therefore, the proposed method II had the lowest absolute relative approximation error and needed fewer iterations to discover the root of the nonlinear equations under consideration. This is explained by the fact that the central difference approximation produces a superior approximation than the forward and backward difference approximations of the first derivative. By comparing the Secant method with the suggested approach, I was also able to acquire the nonlinear equations' roots with fewer iterations within a predefined precision limit.

Therefore, it recommended that the suggested methods I and II be used as suitable alternatives for solving nonlinear equations.

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