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Subring in Graph Theory

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Abstract:

In our study, we delved into the intricacies of graph theory by exploring the properties of subrings within various types of graphs. By focusing on prime graphs and simple graphs, we unraveled the complex relationship between subring-prime graphs. Additionally, we delved into the concept of homomorphism within both simple subring graphs and prime subring graphs, adding depth to our analysis.

Keyword: subring, graph, prime graph, simple graph.

1. Introduction

Euler (1707-1782) [1,5]is credited with delving into graph theory and topology in 1736 by successfully resolving a well-known unresolved problem of his era, known as the Konigsberg Bridge problem. The predicament entailed two islands adjacent to each other and connected to the banks of the Pregel River by seven bridges, depicted in the figure below. The task was to initiate the journey from any of the four landmasses, traverse each bridge exactly once, and conclude at the initial vertex. While one may initially attempt an empirical solution, all efforts are bound to be futile. Euler's contribution to this conundrum was rather paradoxical, as he demonstrated that the problem at hand is indeed insurmountable. Euler's approach involved substituting each landmass with a vertex and each bridge with an edge linking the respective points, thereby constructing a "graph" where the vertices are symbolically linked to the four land areas illustrated in the figure below.

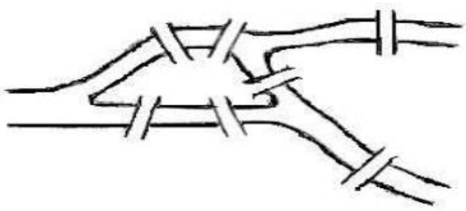

Figure a park in Konigsberg, 1736.

in the current manuscript, the concept of subring is presented within the realm of graph theory along with accompanying results and counterexamples. Additionally, a connection is established between graphs and subrings, leading to a thorough exploration within the field of graph theory. The introduction of the prime graph of a subring marks a significant milestone in this area of study. The term "Prime graph of a subring" is formally defined, and various insights are provided.

2. Preliminaries

Definition 1.1 [1, 4]: assumed finite i.e. $|P| = v$ and $|Q| = u$, $A\ graph\ G = (p, g)$ is P a pair of points and Q a set of edges, and Here $= \{q_1, q_2, \ldots, q_u\}, P(G) = \{p_1, p_2, \ldots, p_v\}$. An edge $q_k = (p_i, p_j)$ is incident with the vertices $p_i\ and\ p_j$. No self-loops or multiple edge is said to be a simple graph.

Definition 1.2 [2]: Let $G = (P(G), Q(G))\ be\ a\ graph\ with\ p$ vertices. A bijection $f : P(G) \rightarrow \{1,2,3, \ldots, p\}\ is\ called\ a\ prime\ labeling\ if\ for\ each\ edge\ q = uv, gcd\ \{f\ (u), f\ (v)\} = 1$. The prime graph is the graph admits prime labeling.

Definition 1.3 [6]: A mathematical system $(R,*,\circ)$ is called ring iff: 1. For any $\partial, \sigma, C \in R\ then\ \partial * (\sigma * C) = (\partial * \sigma) * C$,2. There exist $0 \in R, such\ that\ \partial * 0 = 0 * \partial = \partial, \forall\ \partial \in R,$ 0 is said to by identity element in $R\ w.r.t.*$ 3. For any $\partial \in R\ there\ exist\ b \in R\ such$ that $\partial * \sigma = \sigma * \partial = 0, \sigma$ is said to be invers element of a in $R\ w.r.t * \ (R,*)\ is$ a group , 4. For any $\partial, \sigma, \in R$such that $\partial * \sigma = \sigma * \partial, (R,*)$is a abelian a group 5. For any $\partial, \sigma, C \in R, \partial \circ (\sigma \circ C) = (\partial \circ \sigma) \circ C)$ Is associative, 6. $For\ any\ \partial, \sigma, C \in R\ sit\ \partial \circ (\sigma * C) = (\sigma \circ C) * (C \circ \partial), (R,*,\circ) is\ ring.$

Definition1.4[6]: Let $(R, +,\cdot)$ is a ring anon-empty sub set S of R is said to be sub ring of $R\ iff\ (S, +,.)is$ a ring.

Definition 1.5 [3]: Let R be a ring. $A\ graph\ G(P, Q)$ is $c$alled a $p$rime -graph of R (in short $RG(R)$ if $P = R\ and\ Q = \{St\ sRt = 0\ or\ tRs = 0, and\ S \neq T\}$.

3. Subring of graph theory

Definition 2.1: A graph $G(P, Q)$ can be defined as a graph is said subring- graph if G= S, P=operations first, Q= operation seconds.

Example 2.2: let $G(4, 4)$ be complete graph $= 4, Q = 4$ and $(Z_{4, +, *})$ sub ring, GZ_4 is

said subring-graph.

Definition 2.3: A graph $G(P, Q)$ can be defined as a subring- prime- graph (denoted

by $MG(S)$ if either

$P = s\ and\ Q = \{sSt = 0, s \neq t\ and\ t\ greater\ than\ s = 0\}$, and

$Q = \{sSs = s\ , tSt = t\ and\ s\ greater\ than\ t = t\}$, where S is a subring, M is denoted prime-graph.

Example 2.4: ConsiderZ_{2n}, the subring of $Z_n$the integers modulo Ñ

Let us contains the graph $MG(S)$, where $S = ž = 2n$. We know that

$S = 2n = \{0, 1, 2,3, \ldots, 2n\}. So\ P(MG(S)) = \{0, 1, 2,3, \ldots, 2n\}. Since\ 0S1 = 0,$

$0S2 = 0$, there exists an edge between 0 and 1, and also an edge between 0 and 2. There are no other edge as there are no two non-zero elements s, t ∈ M with $sSt = 0$. So $Q(MG(S)) = \{01, 02, \ldots, 0n\}$.

$1s1 = 1, 2s2 = 2, 3s3 = 3, \dots 2s1 = 2, 3s2 = 3,$ .... is $MG(S)$.

Example 2.5: We ConsiderZ_{12}, the subring $of\ Z_{10}$ integers modulo 10

know that $S = Z$

$$Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7,8,9\}. So$$

$$P\ (MG(S)) = \{0, 1, 2, 3, , 5, 7\}.$$

Now $0S1 = 0, 0S2 = 0, 0S3 = 0, 0S4 = 0, 0S5 = 0,\ 0S6 = 0, 0S7 = 0, 2S1 = 2, 2S3 = o, 2S5 = 0, 7S5 = 5\ , 7S1 = 1\ \ clearly\ Q(MG(S)) = \{1(0), 2(0), 3(0), 7(0), 5(0), , , \check{s} > t = t\}$.

Theorem 2.6: If MG(S) is subring- prime- graph, S is a subring then, neither multiple edge or self-loops n.

proof: assumes that MG(S) is subring-prime-graph from definition subring-prime-graph we get two edges parallel or by using second condition sSs ,tSt is self loops n.

Examples 2.7: The incidence algebra is defined as the algebra over real numbers where for functions f, g element of Inc(G) and x, y element of G, the product fg is a function such that x |f g element of R R| (x) = f(x)g(x) and the sum f+g element of Inc(G) is defined such that (f+g)(x) = f(x) + g(x) for all x element of G. This is easily seen as a subring as all elements in Q are functions from GxG onto R R.

We will now elaborate on various examples of subrings in graph theory. Given a graph $G(P, Q)$, the set of all incidence algebras n times n R R can be a subring of R itself with the operation defined coordinate-wise.

A graph is a diagram with points and lines in which the vertices do not represent algebraic variables and numeric values do not represent a function, but instead the points are coupled with sets of other points. In graph theory, one identifies graphs by an ordered pair G = $G(P, Q)$ where P is a set of vertices and Q is a set of edges. Subrings can be defined in a very similar fashion. This leads to the consideration of a graph as an additive abelian group, where the points are the elements in the group and the lines are created by the element x + y and x for all elements in G. In this essay, we limit graph theory to only the consideration of commutative rings.

4. Simple graphs and Prime Subrings

Firstly, we is giving simple and prime (in terms of $SG(S), MG(S)$) for a subring to is a prime subring.

Theorem 3.1: If S is a Z_n prime -subring of ring Z , n is odd, n=3 only, then the following conditions simple -graph SG(S) is a star- graph. if n even ,n=4 is complete- graph is k_4 .

Proof: suppose that S is a Z_n prime -subring of ring Z , n is odd , n=3 we can represent by complete graph k_3 such that every edge in complete graph of sub ring z_3 not loop and no multiple edge then Z_3 is simple graph (star). If z_4 ,n=4 is square graph not star – graph or greater 4 not star -graph .

Theorem 3.2: Suppose that S is a subring with $o(S) \geq 2$. Then S is prime -subring iff the degree of prime -subring -graph ≥ 2.

Proof: the first direction assume that S is subring of order S ≥ 2 and S is prime –subring S={s_1,s_2,s_3,s_5,s_7,…} since $o(S) \geq 2$, thus prime – subring S ={ s_2,s_3,s_5,s_7,…} .Therefore degree of prime subring–graph≥ 2 . the second direction by same method to proof.

Theorem 3.3: If S is prime subring graph $MG(S)$ homomorphism to $SG(S)$ simple subring graph.

Proof: from definition isomorphic M $G(S) \approx SG(S)$

$$ð: \mathrm{M\ G(S)} \rightarrow \mathrm{SG(S)}$$

$ð(ϰ + Ʊ) = ð(ϰ) + ð(Ʊ)$

$ð(ϰƱ) = ð(ϰ)ð(Ʊ)$

$ϰ, Ʊ \in MG(S),$

$ϰSƱ = 0 such\ that\ ð(ϰƱ) = 0$

$ð(ϰ) + ð(Ʊ) = O + O = ϰSƱ + ϰSƱ$

$If\ ϰSƱ = ϰ\ , ð(ϰ + Ʊ) = ϰ + Ʊ = ϰSƱ + ϰSƱ$

$= MG(S)(ϰ) + SG(S)(Ʊ)$

$= ðϰ) + ð(Ʊ)$

If $ϰSƱ = Ʊ$

$ð(ϰ + Ʊ) = ϰ + Ʊ = ϰSƱ + ƱSϰ$

$= SG(S)(Ʊ) + MG(S)(ϰ)$

$= ð(ϰ) + ð(Ʊ)$

$ϰSƱ = 0\ such\ that\ ð(ϰ + Ʊ) = 0$

$ð(ϰ). ð(Ʊ) = 0.0 = (ϰSƱ). (ƱSϰ)$

$If\ ϰSƱ = ϰ\ , ð(ϰ. Ʊ) = ϰ. Ʊ = (ϰSƱ). (ƱSϰ)$

$= MG(S)(ϰ). SG(S)(Ʊ)$

$= ð(ϰ). ð(Ʊ)$

If $ƱSϰ = Ʊ$

$ð(ϰ. Ʊ) = ϰ. Ʊ = (ϰSƱ). (ƱSϰ)$

$= SG(S)(Ʊ). MG(S)(ϰ)$

$= ð(ϰ). ð(Ʊ)$

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