Advances in Nonlinear Variational Inequalities
ISSN: 1092-910X
Vol 27 No. 2 (2024)

Three Modified Three-Term Conjugate Gradient Method in Non-linear Optimization

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Article History:

Received: *09-02-2024*

Revised: *14-04-2024*

Accepted: *11-05-2024*

Abstract:

In this work several a Tree –Term Conjugate -Gradient (CG) – algorithms are Modified; and that satisfies the sufficient descent condition and a global convergence . primariliy, us- have derive three new methods of this type algorithms . and we compared with (HS) , (PR) and (LS). Algorithms by using (15) Well-Known test functions. And by using Wolfe and Strong Wolfe condition on every iteration. These new algorithms are used different memory less (BFGS) algorithms. Our new numerical results are proved very robust and efficient than the same classes of algorithms.

Keywords: Non_linear,min algorithm,CG,3 Term Conjugate Gradient

1. Introduction

The unconstrained- optimization problem which defined as

$$\text{Min } f(x) \ , \ \ x \in R^n, \tag{1}$$

Where$f: R^n \to R$ is real valued , and continuously differential function .

$\{x_k\}$ is a sequence which generates from a non/linear conjugate gradient method , $k \geq 1$ defined by

$$x_{k+1} = x_k + \alpha_k d_k \tag{2}$$

Where $x_1 \in R^n$ an initial guess. and $a_k = \alpha_k d_k$.

α_k is the step size which obtained by some line searches , a search direction d_k , at the . primariliy iteration is steepest descent direction .defined $d_1 = -g_1$, and defined other search direction by

$$d_{k+1} = -g_{k+1} + \beta_k d_k \qquad \text{[1],[2],[3]} \tag{3}$$

Where $g_k = \nabla f(x_k)$, β_k have many formulas for example:

$$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{d_k^T y_k} \tag{4}$$

$$\beta_k^{CD} = -\frac{g_{k+1}^T g_{k+1}}{d_k^T g_k} \tag{5}$$

$$\beta_k^{PR} = \frac{g_{k+1}^T y_k}{g_k^T g_k} \tag{6}$$

$$\beta_k^{LS} = -\frac{g_{k+1}^T y_k}{d_k^T g_k} \tag{7}$$

Where $y_k = g_{k+1} - g_k$, $\|.\|$ is Euclidean- norm of the vectors .Several methods of general (TTCG)algorithms was proposed by (Beale ,1972) as [4],[5],[6],[7],[8]

$$d_{k+1} = -g_{k+1} + \beta_k d_k + \gamma_k d_k \tag{8}$$

$$\text{for} \quad "" \; \beta_k = \beta_k^{HS}, \beta_k^{FR}, \beta_K^{PR} \ldots\ldots\ldots. "" \tag{9}$$

and $\gamma_k = 0$ where $k = t+1$ and $\gamma_k = \frac{g_{k+1}^T y_k}{d_t^T y_k}$ where $k > t+1$ Many (Three-Term CG) methods was in proved and search directions are as follows

$$d_{K+1}^{TTPR} = -g_{k+1} + \beta_k^{PR} d_k - \theta_k y_k \tag{10}$$

$$d_{K+1}^{TTHS} = -g_{k+1} + \beta_k^{HS} d_k - \theta_k y_k \tag{11}$$

$$d_{K+1}^{TT.Al\ Bayati} = -g_{k+1} + \beta_k^{LS} d_k - \theta_k y_k \tag{12}$$

where different values for θ_k and $\emptyset_k$this work consists of as follows∴ in part- 2, presents three new three terms conjugate gradient methods , & then prove the descent condition and sufficient descent condition and a global convergence for new methods. Last part give some numerical experiments for the new methods and give the conclusion.,[9],[10],[11],[12],[13]

2. New Three-Term(CG) -Methods (TTCG) :

The direction d_{k+1}^{new1} is compute as follows:

$$d_{k+1}^{new1} = -Q_{k+1}^1 \; g_{k+1} \tag{13}$$

$$Q_{k+1}^1 = I - \frac{s_k^T y_k - y_k^T s_k}{y_k^T s_k} + \left[1 + \frac{\|y_k\|^2}{y_k^T s_k}\right] \frac{s_k^T s_k}{y_k^T s_k}$$
(14)

The Memory less (BFGS) method was

$$H_{k+1}^{BFGS} = H_k - \left[\frac{s_k^T y_k H_k + H_k y_k^T s_k}{y_k^T s_k}\right] + \left[1 + \frac{y_k^T H_k y_k}{y_k^T s_k}\right] \frac{s_k^T s_k}{y_k^T s_k} \qquad [14],[15],[16]$$
(15)

We have

$$d_{k+1} = -g_{k+1} + \left[\frac{s_k^T y_k + y_k^T s_k}{y_k^T s_k}\right] g_{k+1} - \left[1 + \frac{\|y_k\|^2}{y_k^T s_k}\right] \frac{s_k^T s_k}{y_k^T s_k} g_{k+1} \tag{16}$$

and after simplified steeps and $H = I$ we are get

$$d_{k+1} = -g_{k+1} - \frac{s_k^T g_{k+1}}{y_k^T s_k} y_k + \frac{z_k^T g_{k+1}}{y_k^T s_k} s_k - \left[1 + \frac{\|y_k\|^2}{y_k^T s_k}\right] \frac{s_k^T g_{k+1}}{y_k^T s_k} s_k \qquad [17]$$
(17)

Since $s_k = \alpha_k d_k$

$$d_{k+1}^{N1} = -g_{k+1} + \frac{g_k^T (z_k - t\, s_k)}{d_k^T y_{k-1}} \, d_k - \frac{s_k^T g_{k+1}}{y_k^T s_k} y_k - \left[1 + \frac{\|y_k\|^2}{y_k^T s_k}\right] \frac{s_k^T g_{k+1}}{y_k^T s_k} s_k$$
(18)

$$d_k = -g_k + \beta_k d_{k-1} \tag{19}$$

$$d_{k+1}^{N1} = -g_{k+1} + \frac{g_k^T(z_k - t\, s_k)}{d_k^T y_{k-1}} d_k + \theta_k(y_k - t_1 s_k) \tag{20}$$

Where

$$\beta^{N1} = \frac{g_k^T(z_k - t\, s_k)}{d_k^T y_{k-1}} \qquad \text{for } t \geq 0 \tag{21}$$

$$\theta_k = -\frac{s_k^T g_{k+1}}{y_k^T s_k} \tag{22}$$

and $t_1 = \left[1 + \frac{\|y_k\|^2}{y_k^T s_k}\right]$ (23)

and $z_k = y_k + \frac{\theta_k}{\|s_k\|^2} s_k$[18],[19],[20] (24)

Also the direction d_{k+1}^{N2} is compute as ∴

$$d_{k+1}^{N2} = -g_{k+1} + \frac{g_k^T(z_k - t\, s_k)}{\|g_{k-1}\|^2}\, d_k + \theta_k(y_k - t_2 s_k) \tag{25}$$

$$\beta^{N2} = \frac{g_k^T(z_k - t\, s_k)}{\|g_{k-1}\|^2} \qquad \text{for } t \geq 0 \tag{26}$$

Where

$$\theta_k = -\frac{s_k^T g_{k+1}}{y_k^T s_k} \tag{27}$$

and $t_2 = \left[1 + 2\frac{\|y_k\|^2}{y_k^T s_k}\right]$ (28)

and $z_k = y_k + \frac{1}{3}\frac{\theta_k}{\|s_k\|^2} s_k$ (29)

Finally , compute search direction d_{k+1}^{N3} as follows

$$d_{k+1} = -Q_{k+1}^2 g_{k+1} \tag{30}$$

$$H_{k+1} = H_k + \left[\frac{2y_k^T H_k y_k}{(s_k^T y_k)^2}\right] s_k s_k^T - \left[\frac{H_k y_k s_k^T + s_k y_k^T H_k}{y_k^T s_k}\right] \tag{31}$$

$$Q_{k+1}^2 = I + \left[\frac{2y_k^T y_k}{(s_k^T y_k)^2}\right] s_k s_k^T - \frac{y_k s_k^T + s_k^T y_k}{s_k^T y_k} \tag{32}$$

$$d_{k+1}^{N3} = -g_{k+1} + \frac{g_k^T(z_k - t_3\, s_k)}{-g_{k+1}^T d_{k-1}}\, d_k + \theta_k(y_k - t_3 s_k) \tag{33}$$

$$\beta^{N3} = \frac{g_k^T(z_k - t\, s_k)}{-g_{k+1}^T d_{k-1}} \qquad \text{for } t \geq 0 \tag{34}$$

Where

$$\theta_k = -\frac{s_k^T g_{k+1}}{y_k^T s_k} \tag{35}$$

and $t_3 = \frac{2\|y_k\|^2}{s_k^T y_k}$ (36)

and $z_k = y_k + \frac{2}{3}\frac{\theta_k}{\|s_k\|^2} s_k$[21],[22] (37)

3. New Three TCG Algorithms

Step 1:Let $x_0 \in R^n$, $0 < \delta < \sigma < L$,$t \geq 0$,$k = 0$, $d_0 = -g_0$

Step2:If Satisfies the Stopping Criteria ($\|g_k\|^2 \leq 10^{-6}$) then stop.

Step3:If $\alpha_k > 0$ satisfies the Wolf Condition ,then compute

$$z_k = x_k + \alpha_k d_k \quad ,y_k = g_k - g_z \,,\; g_z = \nabla f(z)$$

Compute , $u_k = \propto_k g_k^T d_k$,$v_k = -\alpha_k y_k^T d_k$, If $v_k \neq 0$, then

$\lambda_k = -\frac{u_k}{v_k}$, Calculate $x_{k+1} = x_k + \lambda_k \alpha_k d_k$,else

Compute $x_{k+1} = x_k + \alpha_k d_k$

Step4: Calculate (θ_k) and (t) from the equations (23,28,36) and (24,29,37) respectively

Step5: if $|g_{k+1}^T g_k| > 0.2\|g_{k+1}\|^2$ is satisfied then set,

compute the new search direction $d_{k+1}^{New} = -g_{k+1}$, (21),(26),(34)

Step6: set $k = k + 1$, go to step 2

4. Convergence Analysis.

First we have some mild assumptions, that is satisfied the sufficient Descent -property and to proof a global convergence property.

4.1 Proposition 1

Also we have the strong Wolfe conditions as :

$$f(x_k + \alpha_k d_k \leq f(x_k) + \mu \alpha_k g_k^T d_k$$

$$|g_{k-1}^T d_k| \leq -\sigma g_k^T d_k \tag{38}$$

where $0 < \mu < \sigma < 1$

All the search direction d_{k+1}^{New} where given by (21),(26),(34) and of this are sufficient descent direction

$$g_{k+1}^T d_k < 0 \text{ if the line search are satisfies the strong Wolf conditions (38)} \tag{39}$$

Proof:

First, If the new search directions $d_{k+1}^{N1,N2,N3}$ are used and since the line search satisfied Wolf conditions , we have $y_k^T s_k > 0$, then directly we computed

$$g_{k+1}^T d_{k+1}^{N1} = -\|g_{k+1}\|^2 - \left[1 + \frac{\|y_k\|^2}{y_k^T s_k}\right] \frac{\left(s_k^T g_{k+1}\right)^2}{y_k^T s_k} < 0 \tag{40}$$

Also we have

$$g_{k+1}^T d_{k+1}^{N2} = -\|g_{k+1}\|^2 - \left[1 + 2\frac{\|y_k\|^2}{y_k^T s_k}\right] \frac{\left(s_k^T g_{k+1}\right)^2}{y_k^T s_k} < 0 \tag{41}$$

and

$$g_{k+1}^T d_{k+1}^{N3} = -\frac{1}{2}\|g_{k+1}\|^2 < 0 \tag{42}$$

4.2 Proposition 2 (T)

a) IF $S = \{x: x \in R^n, f(x) \leq f(x_0)\}$ bounded , and (x_0) is a start point .

b) For any neighborhood Ω of S , and f is continuously differentiable and the Lipchitz conditions other exists , then there exist - a constant $L \geq 0$ S.T
$\| g(x) - g(x_k)\| \leq L \|x - x_k\|, \forall\ x,\ (x_k) \in \Omega$ and for some assumption of f $\Gamma \geq 0$ exist such that $\|g(x)\| \leq \Gamma$ [23],[24]

4.2.1 Lemma1

If (T, a) and (T ,b) are hold, then

$$\sum_{k=0}^{\infty} -\alpha_k g_k^T d_k < plus\ infint \tag{43}$$

4.2.2 Lemma2

If assume that the assumption (T) (a) and (b) as well as the descent condition hold then

$$\sum_{k=0}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < plus\ infint \tag{44}$$

4.2.3 Lemma3

Assume the part (T , a) and (T , b) are satisfied-, and for any- conjugate gradient Algorithm (38) where d_{k+1} is a descent direction and $\propto_k$ is obtained by the strong Wolfe line search (38) if

$$\sum_{k\geq 1} \frac{1}{\|d_{k-1}\|^2} = plus\ infint \quad [25],[26],[27] \tag{45}$$

4.2.4Theorem :

For a condition (T ,a) and (T , b) hold and satisfied and we have a general optimization algorithms and(33); Where $d_{k+1}^{N1,N2,N3}$ are sufficient descent direction and α_k is computed by the strong Wolfe line search, let *f* is a uniformly convex function on *S* then

$$\lim_{k\to\infty} \|g_k\| = 0 \tag{46}$$

Proof

from the general optimization algorithms and from Lipchitz continuity we have $\|y_k\| \leq L\|s_k\|$ on the other hand from uniform convexity $y_k^T s_k \geq A\|s_k\|^2$ Choosing the Cauchy inequality assumption (T) (a) and (b) and the above inequalities we have

$$|\theta_k| \le \left|\frac{s_k^T s_k}{y_k^T s_k}\right| \le \frac{\|s_k\|^2}{A\|s_k\|^2} \le \frac{1}{A}$$

$$\left|\beta_k^{SPR}\right| \le \left|\frac{\theta_k g_{k+1}^T y_k}{\alpha_k \theta_{k-1} g_k^T g_k}\right| \le \left|\frac{\tau L\|s_k\|}{\|s_k\|^2}\right| \le \frac{\tau L}{\|s_k\|} \tag{47}$$

$$|y_k| \le \frac{|s_k^T g_{k-1}|}{\|g_k\|^2} \le \frac{\tau}{\|s_k\|} \tag{48}$$

Therefore using (47,48) and simplified steeps we get

$$\|d_k\| = |\theta_k|\|g_{k+1}\| + \left|\beta_k^{SPR}\right|\|s_k\| + |y_k|\|\theta_k\|\|y_k\| + |y_k|\|s_k\| \le \frac{\tau}{A} + \tau L + \tau\left[\frac{1}{A}\ L + 1\right] \tag{49}$$

Thus showing that is true By Propositions (1) and (2) it follows that is true which for uniformly convex function is equivalent to (1) and (2)

5. Numerical results

This section provides the numerical results that assessed the effectiveness of the Three term CG algorithms employing the Fletcher-Reeves (FR), Polak-Ribiere (PR), and Liu-Storey (LS) techniques. This information is also available in reference [28].

The program's requirements for stopping $\|g_{k+1}\| \le 10^{-5}$ and written in (FORTRAN90). Total shows the number of the function (NOF) and the number of the iteration (NOI) and confirms that the new methods($DN1$, $DN2$, $DN3$) are superior (NI) with dimension n=1000, 10000[15],[26]

Table(1) Comparison between the new $DN1$, $DN2$, $DN3$ methods against DLS , DPR , DHS methods for the Total of 30-problems with n=1000 , 10000

P.No. Fns	n	DLS		DPR		DHS		$DN3$		$DN2$		$DN1$	
		NI	NF	NI	NF	NI	NF	NI	NF	NI	NF	NI	NF
1	1000	29	44	26	41	26	41	10	32	10	32	10	32
	10000	50	108	50	108	49	105	27	84	24	75	24	75
2	1000	33	56	33	53	29	49	7	23	7	23	7	23
	10000	44	100	48	98	42	95	28	81	28	82	20	75
3	1000	43	78	43	74	34	73	8	20	8	20	5	17
	10000	62	93	62	93	57	89	29	81	29	84	22	70
4	1000	28	49	28	49	28	49	9	30	10	32	8	28
	10000	53	83	53	83	46	73	21	67	21	67	21	67
5	1000	39	86	39	85	38	83	24	69	23	69	23	69
	10000	73	158	73	158	73	155	41	123	41	123	39	120
6	1000	34	58	32	55	28	49	9	30	9	30	8	28
	10000	57	121	56	119	49	109	35	95	35	95	30	89
7	1000	33	55	25	45	25	45	9	25	6	21	6	21
	10000	49	65	44	54	44	54	20	44	18	42	18	42
8	1000	83	166	83	166	77	165	33	100	33	100	32	97
	10000	33	53	33	53	29	49	12	42	10	40	10	40

9	1000	32	45	32	45	28	41	11	20	9	17	7	17
	10000	49	75	49	75	49	75	25	71	24	71	21	67
10	1000	31	48	31	48	29	46	12	32	11	30	10	30
	10000	45	104	45	104	43	101	29	87	29	89	29	87
11	1000	26	45	23	43	23	43	9	29	9	29	6	23
	10000	58	94	58	94	56	94	35	115	35	115	35	115
12	1000	44	104	41	102	39	93	22	71	22	70	23	70
	10000	57	134	53	129	53	123	31	101	31	101	30	99
13	1000	33	55	29	53	27	47	10	30	8	28	8	28
	10000	49	113	47	113	46	109	29	92	29	92	27	90
14	1000	28	96	28	96	25	94	9	43	7	41	7	41
	10000	31	58	29	55	29	55	11	33	11	30	9	28
15	1000	55	106	54	106	54	106	31	92	25	85	25	85
	10000	59	134	54	129	54	129	38	112	38	112	36	109
Total		1343	2584	1301	2526	1228	2439	624	1874	570	1836	556	1682

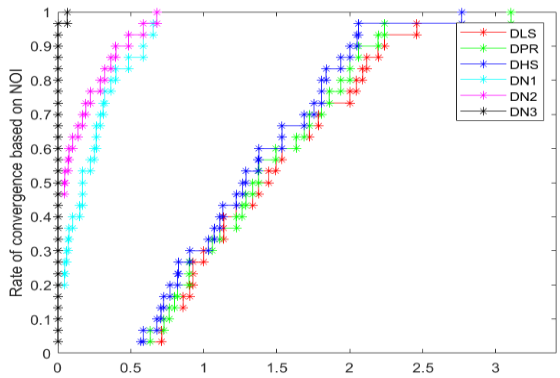

Figure 1: Number of iteration comparing between New methods to standard methods

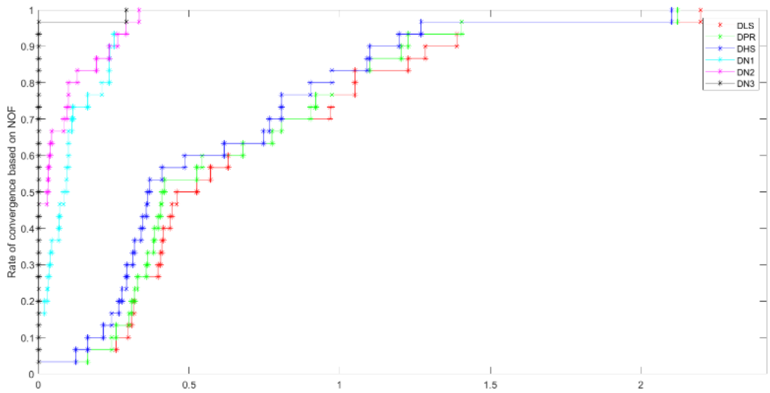

Figure 2: Number of function evaluation comparing between new methods to standard methods

Clearly we have from the Table (2) that New1algorithm Vitim beats (HS) algorithm in about (55%) NOI; (32%) NOF, from Table(3) we have the New2 algorithm beats (PR) algorithm in about (57%) NOI,(28%) NOF ,Also from Table(4)we have the New3algorithm beats (LS) algorithm in about (54%) NOI,(28%) NOF.

Table(2) percentage modified of the new 1 algorithms

	HS algorithm	New 1 algorithm
NOI	100%	45%
NOF	100%	68%

Table (3) percentage modified of the new2 algorithms

	PR algorithm	New 2 algorithm
NOI	100%	43%
NOF	100%	72%

Table (4) percentage modified of the new 3 algorithms

	LS algorithm	New 3 algorithm
NOI	100%	46%
NOF	100%	72%

6. Conclusion

1. The CG Methods are proposed for solving nonlinear optimization Problems
2. Adequate decrease and worldwide convergence can be achieved under certain conditions.

   The numerical findings shown in the previously mentioned figure.

3 .The new algorithms (d^{N1}, d^{N2}, d^{N3}) have prove its efficiency through results in table(1) and(2) and table(3) and table(4)

4. The numerical tests were conducted on problems with low and high dimensionality, with

Appendix

The Test Function For Unconstrained Optimization

No.	**The Test Function**
1	Beale
2	Trigonometric
3	Generalized Quadratic
4	Hager
5	Diagonal 1
6	Diagonal 2
7	EDENSCH
8	EDENSCHNB
9	FLETCHER
10	NONDIA
11	Extended Rosen rock
12	Extended Powell

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