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Theory of Cartan Space with the Generalized Square Metric

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Abstract: The rise of generalized square metrics in Finsler geometry is linked to diverse classifications of (α,β)-metrics, showcasing outstanding geometric properties within this field. In this research paper, we have obtained a necessary and sufficient conditions under which a Cartan space with the generalized square metric, $K(x,\omega) = \frac{[\alpha(x,\omega)+\beta(x,\omega)]^{n+1}}{[\alpha(x,\omega)]^n}$,admitting h-metrical d-connection becomes a locally Minkowski and conformally flat space.

Keywords: Cartan Space, Generalized square metric, Minkowski space, Conformally flat space.

AMS Subject Classification: 53B40, 53C60

1. Introduction

E. Cartan was a French mathematician who invented the theory of Cartan space in 1933 [2]. Cartan space is the dual of a Finsler space [6]. The relation between Cartan space and Finsler space has been studied by F. Brickell [1], H. Rund [10] and others. R. Miron ([6], [7]) introduced the theory of Hamiltonian space, He proved that Cartan space is a particular case of Hamilton space. The notion of (α,β)-metric in Cartan space was introduced by T. Igrashi ([4], [3]. He obtained the metric tensors and some invariants which characterize the special class of Cartan spaces with (α,β)-metric. H.G. Nagaraja [8], G. Shanker [11], M. Rafee [9] and Tripathi [13] have also made significant development in the theory of Cartan spaces with (α,β)-metric. The paper is organized as follows:

In Section 2, we give basic definitions and results required for subsequent sections. In Section 3, we deal with Cartan space with a generalized square (α,β)-metric admitting h-metrical d-connection. In Section 4, we study the conformal change of Cartan space and find some important results.

2. Preliminaries

We recall some important definitions like Finsler metric in cotangent bundle, Cartan space etc:

Definition 2.1 (Finsler Metric of Cotangent Bundle) *Let M be a smooth manifold and T^*M be its cotangent bundle. A C^∞ function $K: T^*M\{0\} \to R$ is called Finsler metric or Finsler fundamental function on the cotangent bundle T^*M if it satisfies the following properties:*

(1) Positivity: $K(x,\omega) \geq 0$ for all $\omega \in T_p^*M$.

(2) Positive Homogeneity: $K(x,\omega)$ is +ve 1-homogeneous on the fibers of the cotangent bundle T^*M, i.e., $K(x,\lambda\omega) = \lambda K(x,\omega)$, $\forall\ \lambda > 0$; for any $x \in M,\ \omega \in T_x^*M$.

(3) Strict Convexity of $K(x,\omega)$: The hessian matrix defined by $g^{ij}(x,\omega) = \frac{1}{2}\frac{\partial^2 K^2}{\partial\omega^i\partial\omega^j}(x,\omega)$ is positive definite for all $(x,\omega) \in T^*M\{0\}$.

Definition 2.2 (Cartan Space)

A differentiable manifold M equipped with a Finsler metric $K(x,\omega)$ defined on the cotangent bundle T^*M is called a Cartan space.

Cartan space is denoted by $C^n = \big(M, K(x,\omega)\big)$, where $K(x,\omega)$ represents norm of the differential one form $\omega \in T_x^*M$ based at any point $x \in M$. The function $K(x,\omega)$ is called the fundamental function and $g^{ij}(x,\omega) = \frac{1}{2}\frac{\partial^2 K^2}{\partial\omega_i\partial\omega_j}(x,\omega)$ is called the fundamental metric tensor of the Cartan space C^n. In Cartan space the metric $K: T^*M \to [0,\infty)$ is defined from cotangent bundle T^*M to non-negative real numbers, so at a point $x \in M$, $K(x,-)$ eats one-form $\omega \in T_p^*M$ and spits non-negative reals, amounts to saying that Cartan space is constructed on the cotangent bundle T^*M in the same way a Finsler space $\big(M, F(x,y)\big)$, where $F: TM \to [0,\infty)$, is constructed on the tangent bundle TM.

Next we define the norm of a differential one form $\omega \in T_p^*M$ in local coordinates or in terms of fundamental metric tensor g^{ij} of the corresponding Cartan space $\big(M, K(x,\omega)\big)$.

Definition 2.3 (Norm of a Differential one Form)

Let $\big(M, K(x,\omega)\big)$ be a Cartan space , where $K(x,\omega)$ is a Finsler metric on the cotangent bundle T^*M. Then the norm of a differential one form $\omega \in T_p^*M$ at any fixed point $x \in M$ is denoted by $K_x(\omega)$ and defined by

$$K_x^2(\omega) = \frac{1}{2}\frac{\partial^2 K^2}{\partial\omega_i\partial\omega_j}(x,\omega)\omega_i\omega_j$$

$$= g^{ij}(x,\omega)\omega_i\omega_j,$$

where $g^{ij}(x,\omega) = \frac{1}{2}\frac{\partial^2 K^2}{\partial\omega_i\partial\omega_j}(x,\omega)$ is the fundamental metric tensor of the Finsler metric $K(x,\omega)$ of cotangent bundle.

Definition 2.4 $(\mathbf{Cartan space with} (\alpha,\beta) - \mathbf{metric})$

If the fundamental function $K(x,\omega)$ of a Cartan space $C^n = \big(M, K(x,\omega)\big)$ is a function of variables $\beta(x,\omega) = \omega_i b^i(x)$, where $a^{ij}(x)$ is a Riemannian metric and $b^i(x)$ is a vector field depending only on x, then C^n is called Cartan space with (α,β)-metric. Here it is to be remarked that $K(x,\omega)$ must satisfy all the conditions imposed on the fundamental function of a Cartan space.

Let us consider a Cartan space $C^n = \big(M, K(x,\omega)\big)$ with a special (α,β)-metric, known as generalized square metric, $K(x,\omega)=\frac{[\alpha(x,\omega)+\beta(x,\omega)]^{n+1}}{[\alpha(x,\omega)]^{n}}$, where $\alpha=\left(a^{ij}(x,\omega)\omega_i\omega_j\right)^{\frac{1}{2}}$ and $\beta=\omega_i b^i(x)$.

The fundamental tensor $g^{ij}(x,\omega)$ and its reciprocal tensor $g_{ij}(x,\omega)$ of the Cartan space $C^n=\left(M,K(\alpha,\beta)\right)$ are given by [3]

$$g^{ij}=\rho a^{ij}+\rho_0 b^i b^j+\rho_{-1}\left(b^i\omega_j+b^j\omega_i\right)+\rho_{-2}\omega_i\omega_j, \tag{1}$$

where ρ, ρ_0, ρ_{-1} and ρ_{-2} are invariants which are defined and calculated as follows:

$$\rho=\frac{1}{2\alpha}K_\alpha$$

$$=\frac{(\alpha-n\beta)(\alpha+\beta)^n}{2\alpha^{n+2}}$$

$$\rho_0=\frac{1}{2}K_{\beta\beta}$$

$$=\frac{n(n+1)(\alpha+\beta)^{n-1}}{2\alpha^n}$$

$$\rho_{-1}=\frac{1}{2\alpha}K_{\alpha\beta}$$

$$=-\frac{n(n+1)\beta(\alpha+\beta)^{n-1}}{2\alpha^{n+2}}$$

$$\rho_{-2}=\frac{1}{2\alpha^2}\left(K_{\alpha\alpha}-\frac{1}{\alpha}K_\alpha\right)$$

$$=\frac{(\alpha+\beta)^{n-1}\{n(n+1)\beta^2-(\alpha-n\beta)(\alpha+\beta)\}}{2\alpha^{n+4}}$$

and

$$g_{ij}=\sigma a_{ij}-\sigma_0 b_i b_j+\sigma_{-1}\left(b_i\omega_j+b_j\omega_i\right)+\sigma_{-2}\omega_i\omega_j, \tag{2}$$

where

$$\sigma=\frac{1}{\rho}$$

$$=\frac{2\alpha^{n+2}}{(\alpha-n\beta)(\alpha+\beta)^n}$$

$$\sigma_0=\frac{\rho_0}{\rho\tau}$$

$$\tau=\sigma+\sigma_0 B^2+\rho_{-1}\beta$$

$$\sigma_{-1}=\frac{\rho_{-1}}{\rho\tau}$$

$$\sigma_{-2}=\frac{\rho_{-2}}{\rho\tau},$$

where $B^2=b^i b_j$ and B represents the norm of the differential form $\beta(x,\omega)=\omega_i b^i(x)\in T_p^*M$.

The Cartan torsion tensor C^{ijk} [5] is given by

$$C^{ijk} = -\frac{1}{2}\left[r_{-1}b^i b^j b^k + \{\rho_{-1}a^{ij}b^k + \rho_{-2}a^{ij}\omega^k + r_{-2}b^i b^j \omega^k + r_{-3}b^i\omega^j\omega^k + i|j|k\} + r_{-4}\omega^i\omega^j\omega^k\right], \quad (3)$$

where its coefficients r_{-1}, r_{-2}, r_{-3} and r_{-4} are defined and calculated as follows:

$$r_{-1} = \frac{1}{2}K_{\beta\beta\beta}$$
$$= \frac{(n-1)n(n+1)(\alpha+\beta)^{n-2}}{2\alpha^n}$$
$$r_{-2} = \frac{1}{2\alpha}K_{\alpha\beta\beta}$$
$$= -\frac{n(n+1)(\alpha+n\beta)(\alpha+\beta)^{n-2}}{2\alpha^{n+1}}$$
$$r_{-3} = \frac{1}{2\alpha^2}\left(K_{\alpha\alpha\beta} - \frac{1}{\alpha}K_{\alpha\beta}\right)$$
$$= \frac{n(n+1)\beta(\alpha+\beta)^{n-2}(3\alpha+2\beta+n\beta)}{2\alpha^{n+4}}$$
$$r_{-4} = \frac{1}{2\alpha^3}\left(K_{\alpha\alpha\alpha} - \frac{3}{\alpha}K_{\alpha\alpha} + \frac{3}{\alpha^2}K_{\alpha}\right)$$
$$= \frac{n(n+1)\beta^2(\alpha+\beta)^{n-1}\{(n-1)(\alpha+\beta)^{-1}\alpha^3-(n+2)\}}{\alpha^{n+5}}.$$

Let ':' denote the covariant differentiation with respect to Christoffel symbols γ^i_{jk} constructed from a_{ij}. Whenever we talk about Christoffel symbols γ^i_{jk} constructed from a_{ij}, we mean $\gamma^i_{jk} = \frac{1}{2}a^{li}\left(\frac{\partial a_{kl}}{\partial x^j} + \frac{\partial a^{lj}}{\partial x^k} - \frac{\partial a^{jk}}{\partial x^l}\right)$. Since $\omega_{i:k} = 0$ and from Ricci's theorem of tensor calculus [14] we have $a^{ij}_{:k} = 0$, if $b^i_{:k} = 0$, then $g^{ij}_{:k} = 0$. Also, let $\Gamma^i_{jk}(p) = \frac{1}{2}g^{ir}\left(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk}\right)$ be the Christoffel symbols constructed from fundamental metric tensor $g_{ij}(x,\omega)$ of the Cartan space $\left(M, K(x,\omega)\right)$. Now, for the Cartan space $\left(M, K(x,\omega)\right)$, we state canonical d-connection is a triplet given by

$$D\Gamma = \left(N_{jk}, H^i_{jk}, C^{jk}_i\right),$$

where

$$N_{ij} = \Gamma^k_{ij}\omega_k - \frac{1}{2}\Gamma^k_{hr}\omega_k\omega^r\dot{\partial}^h g_{ij} \quad (4)$$

$$H^i_{jk} = \frac{1}{2}g^{ir}\left(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk}\right) \quad (5)$$

$$C^{jk}_i(x,\omega) = -\frac{1}{2}g_{ir}(x,\omega)\frac{\partial g^{jk}(x,\omega)}{\partial\omega^r} = g_{ir}(x,\omega)C^{rjk}(x,\omega) \quad (6)$$

are respectively called canonical N-connection, Christoffel symbols and d-tensor field of type (2,1).

Let h-covariant derivative with respect to $D\Gamma$ be denoted by the symbol $'|_{h}'$. Then, we have the following definition for later use.

Definition 2.5 ([**9**])An h-metrical d-connection on a Cartan space $C^n = (M, K(\alpha(x,\omega), \beta(\omega))$ with (α, β)-metric is a d-connection, $D\Gamma$ on C^n, satisfying the following properties:

(i) h-deflection tensor $D_{ij}(= \omega_{i|j}) = 0$

(ii) $a^{ij}_{|h} = 0$

(iii) $g^{ij}_{|h} = 0$.

3. Cartan spaces with a generalized square metric admitting h-metrical d-connection

In this section we impose a condition on d-connection $D\Gamma$ of the Cartan space with generalized square metric to be h-metrical and in consequence we assess what shapes the corresponding Cartan space assumes.

First we take the h-covariant derivative of generalized square metric:

$$K(x,\omega) = \frac{[\alpha(x,\omega)+\beta(x,\omega)]^{n+1}}{[\alpha(x,\omega)]^n}$$

$$g^{ij}\left(\omega_i\omega_{j|h} + \omega_j\omega_{i|h}\right) + \omega_i\omega_j g^{ij}_{|h} = \frac{(n+1)\alpha^n(\alpha+\beta)^n(\alpha_{|h}+\beta_{|h}) - n(\alpha+\beta)^{n+1}\alpha_{|h}}{\alpha^{2n}}$$

As we have stipulated the d-connection $D\Gamma$ of the Cartan space is h-metrical, therefore by Definition 2.5, we have

$$\omega_{j|h} = 0, \omega_{i|h} = 0, \alpha_{|h} = 0, g^{ij}_{|h} = 0$$

Using these values in above expression, we get

$$g^{ij}\left(\omega_i \times 0 + \omega_j \times 0\right) + \omega_i\omega_j \times 0 = \frac{(n+1)\alpha^n(\alpha+\beta)^n(0+\beta_{|h}) - n(\alpha+\beta)^{n+1} \times 0}{\alpha^{2n}}$$

$$0 = \frac{(n+1)\alpha^n(\alpha+\beta)^n\beta_{|h}}{\alpha^{2n}}$$

$$\beta_{|h} = 0 \qquad (\because \alpha \neq 0, \beta \neq 0) \tag{7}$$

$$(\omega_i b^i(x))_{|h} = 0 \qquad (\because \beta(x,\omega) = \omega_i b^i(x))$$

$$\omega_i b^i(x)_{|h} + b^i(x)\omega_{i|h} = 0$$

As we have stipulated the d-connection $D\Gamma$ of the Cartan space is h-metrical, therefore by Definition 2.5, we have

$$\omega_{i|h} = 0,$$

Using these values in above expression, we get

$$\omega_i b^i(x)_{|h} + b^i(x) \times 0 = 0$$

$$\omega_i b^i(x)_{|h} = 0$$

$$b^i(x)_{|h} = 0 \tag{8}$$

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Now, we find h-covariant derivatives of the coefficients of metric tensor g^{ij} and then use conditions of h-metrical d-connection $D\Gamma$ of Cartan space as follows, we get

$$\because \rho = \frac{(\alpha-n\beta)(\alpha+\beta)^n}{2\alpha^{n+2}}$$

$$\therefore \rho_{|h} = \frac{\alpha^{n+2}(\alpha+\beta)^n\{(\alpha-n\beta)n(\alpha+\beta)^{-1}(\alpha_{|h}-\beta_{|h})+(\alpha_{|h}-n\beta_{|h})\}-(\alpha-n\beta)(n+2)\alpha^{n-1}\alpha_{|h}}{2\alpha^{2n+4}}$$

$$\rho_{|h} = 0. \tag{9}$$

$$\because \rho_0 = \frac{n(n+1)(\alpha+\beta)^{n-1}}{2\alpha^n}$$

$$\therefore \rho_{0|h} = \frac{n(n+1)\{(n-1)\alpha^n(\alpha+\beta)^{n-2}(\alpha_{|h}+\beta_{|h})-2n\alpha^{2n-1}(\alpha+\beta)^{n-1}\}}{2\alpha^{2n}}$$

$$\rho_{0|h} = 0. \tag{10}$$

$$\because \rho_{-1} = -\frac{n(n+1)\beta(\alpha+\beta)^{n-1}}{2\alpha^{n+2}}$$

$$\therefore \rho_{-1|h} = n(n+1)(\alpha+\beta)^{n-1}\frac{[\alpha^{n+2}\{(n-1)\beta(\alpha+\beta)^{n-2}(\alpha_{|h}+\beta_{|h})+\beta_{|h}\}-(n+2)\beta\alpha^{n-1}\alpha_{|h}]}{2\alpha^{2n+4}}$$

$$\rho_{-1|h} = 0. \tag{11}$$

$$\because \rho_{-2} = \frac{(\alpha+\beta)^{n-1}\{n(n+1)\beta^2-(\alpha-n\beta)(\alpha+\beta)\}}{2\alpha^{n+4}}$$

$$\therefore \rho_{-2|h} = 0. \tag{12}$$

The h-covariant differentiation of the equation (1) gives

$$g^{ij}_{|h} = \rho a^{ij}_{|h} + a^{ij}\rho_{|h} + \rho_0(b^i b^j)_{|h} + b^i b^j \rho_0 + \rho_{-1}(b^i\omega^j + b^j\omega^i)_{|h} +$$
$$(b^i\omega^j + b^j\omega^i)\rho_{-1|h} + \rho_{-2}(\omega^i\omega^j)_{|h} + \omega^i\omega^j\rho_{-2|h}$$

$$g^{ij}_{|h} = \rho a^{ij}_{|h} + a^{ij}\rho_{|h} + \rho_0\left(b^i b^j_{|h} + b^j b^i_{|h}\right) + b^i b^j \rho_{0|h} + \rho_{-1}\left(b^i\omega^j_{|h} + \omega^i b^i_{|h} + b^j\omega^i_{|h} + \omega^i b^j_{h}\right)$$
$$\rho_{-1|h}\left(b^i\omega^j + b^j\omega^i\right) + \rho_{-2|h}\left(\omega^i\omega^j_{|h} + \omega^j\omega^i_{|h}\right) + \omega^i\omega^j\rho_{-2|h}$$

Using the conditions of h-metrical d-connection $D\Gamma$ of Cartan space and Equations (8), (9), (10), (11) and (12), above equation reduces to

$$g^{ij}_{|h} = 0.$$

Thus, allowing d-connection $D\Gamma$ of the Cartan space to be h-metrical, it gives two important quantities namely $a^{ij}_{|h} = 0$ (by definition of h-metrical d-connection) and $g^{ij}_{|h} = 0$, i.e., h-covariant derivatives of fundamental metric tensors of associated Riemannian space and Cartan space vanishes.

Now, since $a^{ij}_{|h} = 0$ and $g^{ij}_{|h} = 0$, therefore there corresponding Chritoffel symbols will also be same, i.e., $H^i_{jh} = \gamma^i_{jh}$ and its equivalent condition is given by

$$b^i_{:k} = 0 \tag{13}$$

Now, since $H^i_{jh} = \gamma^i_{jh}$ therefore the curvature tensor D^i_{hjk} of $D\Gamma$ coincides with the curvature tensor R^i_{hjk} of Riemannian connection $R\Gamma = \left(\gamma^i_{jk}, \gamma^i_{jk} y_i, 0\right)$, i.e., $D^i_{hjk} = R^i_{hjk}$.

If the Riemannian curvature tensor vanishes, i.e., $R^i_{hjk} = 0$, the curvature tensor of d-connection also vanishes, i.e., $D^i_{hjk} = 0$. This discussion can be summarized as follows:

Proposition 3.1 *A Cartan space C^n with generalized square metric, $K(x, \omega) = \frac{[\alpha(x,\omega)+\beta(x,\omega)]^{n+1}}{[\alpha(x,\omega)]^n}$, admitting a h-metrical d-connection is locally flat if and only if the associated Riemmanian space is locally flat.*

Now, we find h-covariant derivatives of the coefficients of Cartan torsion tensor C^{ijk} and then use conditions of h-metrical d-connection $D\Gamma$ of Cartan space and equation (7) as follows:

$$\because r_{-1} = \frac{(n-1)n(n+1)(\alpha+\beta)^{n-2}}{2\alpha^n}$$

$$\therefore r_{-1|h} = 0 \tag{14}$$

$$\because r_{-2} = -\frac{n(n+1)(\alpha+n\beta)(\alpha+\beta)^{n-2}}{2\alpha^{n+1}}$$

$$\therefore r_{-2|h} = 0 \tag{15}$$

$$\because r_{-3} = \frac{n(n+1)\beta(\alpha+\beta)^{n-2}(3\alpha+2\beta+n\beta)}{2\alpha^{n+4}}$$

$$\therefore r_{-3|h} = 0 \tag{16}$$

$$\because r_{-4} = \frac{n(n+1)\beta^2(\alpha+\beta)^{n-1}\{(n-1)(\alpha+\beta)^{-1}\alpha^3-(n+2)\}}{\alpha^{n+5}}$$

$$\therefore r_{-4|h} = 0 \tag{17}$$

Now we calculate the value of h-covariant derivative of d-tensor field C^{jk}_i of type (2,1) under the assumption of h-metrical d-connection as follows:

$$\because C^{ij}_k = g_{kr} C^{rij}$$

$$\therefore C^{ij}_{k|h} = \left(g_{kr} C^{rij}\right)_{|h}$$

$$= g_{kr} \times C^{rij}_{|h} + C^{rij} \times 0 g_{kr|h}$$

$$= g_{kr} C^{rij}_{|h}$$

$$= -g_{kr} \frac{1}{2} [r_{-1} b^r b^i b^j + r_{-2} b^r b^i \omega^j + r_{-3} b^r \omega^i \omega^j + r_{-4} \omega^r \omega^i \omega^j + \rho_{-1} a^{ri} b^j +$$

$$\rho_{-2} a^{ri} \omega^j + r|i|j]_{|h}$$

$$= -g_{kr} \frac{1}{2} [r_{-1} \times (b^r b^i b^j)_{|h} + b^r b^i b^j \times 0 r_{-1|h} + r_{-2} \times (b^r b^i \omega^j)_{|h} + b^r b^i \omega^j \times$$

$$0 r_{-2|h} + r_{-3} \times (b^r \omega^i \omega^j)_{|h} + b^r \omega^i \omega^j \times 0 r_{-3|h} + r_{-4} \times (\omega^r \omega^i \omega^j)_{|h} + \omega^r \omega^i \omega^j \times$$

$0r_{-4|h} + \rho_{-1} \times (a^{ri}b^j)_{|h} + a^{ri}b^j \times 0\rho_{-1|h} + \rho_{-2} \times (a^{ri}\omega^j)_{|h} + a^{ri}\omega^j \times$

$0\rho_{-2|h} + \big(r|i|j)_{|h}\big]$

$= -g_{kr}\frac{1}{2}[r_{-1}(b^r b^i b^j)_{|h} + r_{-2}(b^r b^i \omega^j)_{|h} + r_{-3}(b^r \omega^i \omega^j)_{|h} + r_{-4}(\omega^r \omega^i \omega^j)_{|h} +$

$\rho_{-1}(a^{ri}b^j)_{|h} + \rho_{-2}(a^{ri}\omega^j)_{|h} + (r|i|j)_{|h}]$

$$= -g_{kr}\frac{1}{2}[r_{-1}(b^r b^i 0b^j_{|h} + b^r b^j 0b^i_{|h} + b^i b^j 0b^r_{|h}) + r_{-2}(b^r b^i 0\omega^j_{|h} + b^r \omega^j 0b^i_{|h} + b^i \omega^j 0b^r_{|h}) +$$

$$r_{-3}\Big(b^r \omega^i 0\omega^j_{|h} + b^r \omega^j 0\omega^i_{|h} + \omega^i \omega^j 0b^r_{|h}\Big) + r_{-4}\Big(\omega^r \omega^i 0\omega^j_{|h} + \omega^r \omega^j 0\omega^i_{|h} + \omega^i \omega^j 0\omega^r_{|h}\Big) +$$

$$\rho_{-1}\Big(a^{ri}0b^j_{|h} + b^j 0a^{ri}_{|h}\Big) + \rho_{-2}\Big(a^{ri}0\omega^j_{|h} + \omega^j 0a^{ri}_{|h}\Big) + 0\big(r|i|j)_{|h}\big]$$

$$C^{ij}_{k|h} = 0 \qquad (18)$$

One knows that a Cartan space C^n is Berwald space if and only if $C^{ij}_{k|h} = 0$ [12]. Hence from Equation (18), we have the following proposition:

Proposition 3.2 *A Cartan space C^n with generalized square metric, $K(x,\omega) = \frac{[\alpha(x,\omega)+\beta(x,\omega)]^{n+1}}{[\alpha(x,\omega)]^n}$, admitting h-metrical d-connection is a Berwald space.*

In [12], it is deduced that a locally Minkowski space is a Berwald space in which curvature tensor vanishes. Hence, from the Propositions 3.1 and 3.2, we have following theorem:

Theorem 3.3 *A Cartan space with generalized square metric, $K(x,\omega) = \frac{[\alpha(x,\omega)+\beta(x,\omega)]^{n+1}}{[\alpha(x,\omega)]^n}$, admitting h-metrical d-connection is locally Minkowski space if and only if the associated Riemannian space is locally flat.*

4. Conformal change of Cartan space with generalized square metric

In this section our aim is to conformally transform a Cartan space $\big(M, K(x,\omega)\big)$ to another Cartan space $\Big(M, \widetilde{K}(x,\omega)\Big)$ and then to determine the nature of curvature tensor $\widetilde{D}^i_{hjk}$ in the conformally transformed space $\Big(M, \widetilde{K}(x,\omega)\Big)$ under the influence of h-metrical d-connection on the original Cartan space $\big(M, K(x,\omega)\big)$. That is, we are going to determine the shape of conformally transformed space $\Big(M, \widetilde{K}(x,\omega)\Big)$ under the stipulation of h-metrical d-connection on $\big(M, K(x,\omega)\big)$.

For that, consider an n-dimensional Cartan space $C^n = \big(M^n, K(x,\omega)\big)$ equipped with a real smooth n-manifold M and generalized square metric $K(\alpha,\beta)$, given by $K(x,\omega) = \frac{[\alpha(x,\omega)+\beta(x,\omega)]^{n+1}}{[\alpha(x,\omega)]^n}$, where $\alpha = (a^{ij}(x,\omega)\omega_i\omega_j)^{\frac{1}{2}}$ and $\beta = \omega_i b^i(x)$. By a conformal change $\sigma: K \to \widetilde{K}$ such that $\widetilde{K}\big(\tilde{\alpha},\tilde{\beta}\big) = e^{\sigma}K(\alpha,\beta)$, we have the another Cartan space $\tilde{C}^n = \Big(M, \widetilde{K}\big(\tilde{\alpha},\tilde{\beta}\big)\Big)$, where $\tilde{\alpha} = e^{\sigma}\alpha$ and $\tilde{\beta} = e^{\sigma}\beta$.

Putting $\alpha = (a^{ij}(x,\omega)\omega_i\omega_j)^{\frac{1}{2}}$ and $\beta = \omega_i b^i(x)$ in the above relations, we get

$$\tilde{\alpha} = e^{\sigma}\alpha$$

$$\tilde{\alpha} = e^{\sigma}(a^{ij}(x,\omega)\omega_i\omega_j)^{\frac{1}{2}}$$

$$\tilde{\alpha} = (\underline{e^{2\sigma}a^{ij}(x,\omega)}\omega_i\omega_j)^{\frac{1}{2}}$$

$$\tilde{\alpha} = (\underline{\tilde{a}^{ij}}\omega_i\omega_j)^{\frac{1}{2}}$$

$$\tilde{a}^{ij} = e^{2\sigma}a^{ij}(x,\omega)$$

and

$$\tilde{\beta} = e^{\sigma}\beta$$

$$\tilde{\beta} = e^{\sigma}\omega_i b^i(x)$$

$$\tilde{\beta} = \omega_i\underline{e^{\sigma}b^i(x)}$$

$$\tilde{\beta} = \omega_i\underline{b^i}$$

$$\tilde{b}^i = e^{\sigma}b^i(x).$$

Now we calculate the Christoffel symbols $\tilde{\gamma}^p_{rk}$ in conformally transformed space $\left(M,\tilde{K}(x,\omega)\right)$ as follows:

We know from Riemannian geometry Christoffel symbols of second kind γ^p_{rk} from fundamental metric tensor $a^{pq}(x,\omega)$ can be defined as

$$\gamma^p_{qk} = \frac{1}{2}a^{lp}\left(\frac{\partial a_{kl}}{\partial x^q} + \frac{\partial a^{lq}}{\partial x^k} - \frac{\partial a^{qk}}{\partial x^l}\right)$$

Similarly, we can also define the Christoffel symbols $\tilde{\gamma}^p_{rk}$ in conformally transformed space $\left(M,\tilde{K}(x,\omega)\right)$ as

$$\tilde{\gamma}^p_{qk} = \frac{1}{2}\tilde{a}^{lp}\left(\frac{\partial \tilde{a}_{kl}}{\partial x^q} + \frac{\partial \tilde{a}_{lq}}{\partial x^k} - \frac{\partial \tilde{a}_{qk}}{\partial x^l}\right)$$

$$= \frac{1}{2}e^{2\sigma}a^{lp}(x,\omega)\left(\frac{\partial e^{2\sigma}a_{kl}(x,\omega)}{\partial x^q} + \frac{\partial e^{2\sigma}a_{lq}(x,\omega)}{\partial x^k} - \frac{\partial e^{2\sigma}a_{qk}(x,\omega)}{\partial x^l}\right)$$

$$= \frac{1}{2}e^{2\sigma}a^{lp}\left[\left(e^{2\sigma}\frac{\partial a_{kl}}{\partial x^q} + a_{kl}\frac{\partial e^{2\sigma}}{\partial x^q}\right) + \left(e^{2\sigma}\frac{\partial a_{lq}}{\partial x^k} + a_{lq}\frac{\partial e^{2\sigma}}{\partial x^k}\right) - \left(e^{2\sigma}\frac{\partial a_{jq}}{\partial x^l} + a_{qk}\frac{\partial e^{2\sigma}}{\partial x^l}\right)\right]$$

$$= \frac{1}{2}e^{2\sigma}a^{lp}\left[\left(e^{2\sigma}\frac{\partial a_{kl}}{\partial x^q} + 2e^{2\sigma}a_{kl}\frac{\partial \sigma}{\partial x^q}\right) + \left(e^{2\sigma}\frac{\partial a_{lq}}{\partial x^k} + 2e^{2\sigma}a_{lq}\frac{\partial \sigma}{\partial x^k}\right) - \left(e^{2\sigma}\frac{\partial a_{qk}}{\partial x^l} + 2e^{2\sigma}a_{qk}\frac{\partial \sigma}{\partial x^l}\right)\right]$$

$$= \frac{1}{2}e^{4\sigma}a^{lp}\left[\left(\frac{\partial a_{kl}}{\partial x^q} + \frac{\partial a_{lq}}{\partial x^k} - \frac{\partial a_{qk}}{\partial x^l}\right) + \left(2a_{kl}\frac{\partial \sigma}{\partial x^q} + 2a_{lq}\frac{\partial \sigma}{\partial x^k} - 2a_{qk}\frac{\partial \sigma}{\partial x^l}\right)\right]$$

$$= e^{4\sigma}\left[\frac{1}{2}a^{lp}\left(\frac{\partial a_{kl}}{\partial x^q} + \frac{\partial a_{lq}}{\partial x^k} - \frac{\partial a_{qk}}{\partial x^l}\right) + \left(a^{lp}a_{kl}\sigma_q + a^{lp}a_{lq}\sigma_k - a^{lp}a_{qk}\sigma_l\right)\right]$$

$$= e^{4\sigma}\left[\gamma^p_{qk} + \left(\delta^p_k\sigma_q + \delta^p_q\sigma_k - a_{qk}\sigma^i\right)\right]$$

Hence, the components of Christoffel symbols $\tilde{\gamma}^p_{qk}$, constructed from $\tilde{a}^{pq}$, in conformally transformed space are given by

$$\tilde{\gamma}^p_{qk} = \gamma^p_{qk} + B^p_{qk}, \qquad (19)$$

where $B^p_{qk} = \sigma_k \delta^p_q + \sigma_q \delta^p_k - a_{kq}\sigma^p$, $\sigma^p = \sigma_q a^{pq}$.

The covariant derivative of $\tilde{b}^p$ with respect to $\tilde{\gamma}^p_{rk}$, yields

$$\tilde{b}^p_{:k} = e^{\sigma}\left(b^p_{:k} + 2\sigma_k b^p + b^r \sigma_r \delta^p_k - \sigma_p b^r a_{rk}\right). \qquad (20)$$

Transvecting the Equation (20) by $\tilde{b}^k$, and putting

$$M^p = \frac{1}{B^2}\left(b^k b^p_{:k} - \frac{b^r_{:r} b^p}{n+4}\right), \qquad (21)$$

we have $\sigma^p = \widetilde{M}^p - M^p$, from which we get $\sigma_p = \widetilde{M}^p - M_p$. Substituting the values of σ_p and σ^p in Equation (19) and using $D^p_{hq} = \gamma^p_{hq} + \delta^p_h M_q + \delta^p_h M_q + \delta^p_q M_h - M^p a_{hq}$, we find

$$\widetilde{D}^p_{hq} = D^p_{hq}. \qquad (22)$$

Here D^p_{hq} is a symmetric and conformally invariant linear connection on M.

The whole discussion can be summarized in the following proposition.

Proposition 4.1 *Let $C^n = (M, K(,\omega))$ be a Cartan space with generalized square metric $K(x,\omega) = \frac{[\alpha(x,\omega)+\beta(x,\omega)]^{n+1}}{[\alpha(x,\omega)]^n}$. Then, there exists a conformally invariant symmetric linear connection D^p_{qk} on M.*

Next, if we denote the curvature tensor of D^p_{qk} by D^p_{hqk}, then from the Equation (22), we get

$$\widetilde{D}^p_{hqk} = D^p_{hqk}. \qquad (23)$$

Since $b^p_{:k} = 0$, from Equation (21), we get $M^i = 0$. Hence, we deduce that $D^p_{qk} = \gamma^p_{qk}$ and $D^p_{hqk} = R^p_{hqk}$.

Thus we have the following proposition.

Proposition 4.2 *Let $C^n = (M, K)$ be a Cartan space with generalized square metric, $K(x,\omega) = \frac{[\alpha(x,\omega)+\beta(x,\omega)]^{n+1}}{[\alpha(x,\omega)]^n}$, admitting h-metrical d-connection. Then, there exists a conformally invariant symmetric linear connection D^p_{qh} such that $D^p_{qk} = \gamma^p_{qk}$ and it's curvature tensor $D^p_{hqk} = R^p_{hqk}$.*

Next, if the associated Riemannian space (M, α) is locally flat, that is, $R^p_{hqk} = 0$, then from Proposition 4.2 and Equation (23), we deduce that $\widetilde{D}^p_{hqk} = 0$, that is, the space C^n is conformally flat. Thus we have the following theorem.

Theorem 4.3 *Let $C^n = (M, K)$ be a Cartan space withgeneralized square metric, $K(x,\omega) = \frac{[\alpha(x,\omega)+\beta(x,\omega)]^{n+1}}{[\alpha(x,\omega)]^n}$, admitting h-metrical d-connection. Then the space C^n is conformally flat if and* *only if the associated Riemannian space is locally flat.*

Conclusion: Now after all, one may ask why authors of the article are so interested to carry forward this theory of Cartan space with the generalized square metric $K(x,\omega) = \frac{[\alpha(x,\omega)+\beta(x,\omega)]^{n+1}}{[\alpha(x,\omega)]^{n}}$, where $n = 1,2,3,\ldots$. My answer is very assertive that generalization of any theory is always fascinating due to its nature to bring various special cases under one umbrella. For example, the theorems that we have proved works for every natural number $n \in N$.

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