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Connectedness in New Pseudo-Topological Vector Spaces

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Abstract: Studying connectedness is an important topic in classical topology, so we sought to study this property in the new pseudo topological vector space (T_(p ̃VS)) which is built on the convergence of the filters.

Keywords: Filter, Pseudo topological and Connected Spaces.

1. Introduction

Frölicher, A. and Walter B presented a definition of the pseudo topological vector space. Furthermore, Averbuch in [2] and Gähler et al. (1976) gave several other definitions of pseudo topological vector space. Several work was included a study of the topological properties such as connectedness, disconnectedness, and continuity. Due to its importance in paving the way for applied studies, in this paper, we decided to study the connectedness of the properties of the psudo topological vector space that we presented in [5].

2.Preliminaries

Definition 1[1]: Let Ξ is a vector space over R, a filter in Ξ is a nonempty set £ such that: 1- $M \in £$ and $M \subset N \Rightarrow N \in £$; 2- $M, N \in £ \Rightarrow M \cap N \in £$ 3-$\emptyset \notin £$.
And β is called filter basis if satisfies:1- $\emptyset \notin \beta$; 2- $\forall £, \rho \in \beta$ there exists $\sigma \in \beta$ such that $£ \cap \rho \supset \sigma$ is a filter basis.
And if $w \in \Xi, [w] = \{M \subseteq \Xi: w \in M\}$ is a filter.
Definition 2[1]: Let $\mathcal{F}(\Xi)$ be a family of all filters in Ξ. Then $\tau : \Xi \to 2^{\mathcal{F}(\Xi)}$ is called a psudo-topology τ on Ξ which is if $£ \in \tau(w)$, we say that the filter £ converges to w, we write $£ \in \tau(w)$ (Or $£ \downarrow_w$), the conditions hold $\forall$ filters £ and p:$[w] \in \tau(w)$;$£ \in \tau(w)$, $£ \subseteq p \Rightarrow p \in \tau(w)$;$£ \in \tau(w), p \in \tau(w)) \Rightarrow £ \cap p \in \tau(w)$.
Definition 3[5]: A vector space together with a compatible pseudo topology topology (The two functions addition and multiplication are continuous) on it is said to be a pseudo-topological vector space (briefly, $T_{\tilde{p}VS}$).
Definition 4[5]: Let $\mathcal{A} \subseteq \Xi$, we said it open set if $\mathcal{A} = int(\mathcal{A})$, such that
$int(\mathcal{A}) = \{w \in \mathcal{A}: £ \in \tau(w) , \mathcal{A} \in £\}$.
Definition 5[5]: $\mathcal{A}$ is closed set if $\mathcal{A} = \bar{\mathcal{A}} =$(CL $\mathcal{A}) = \{w \in \Xi : \exists p \downarrow_w \Xi$ and $\mathcal{A} \in p\}$.
Definition 6[2]: $\mathcal{N}_\tau(w)$ is neighborhood filter of w ∈ Ξ:
$\mathcal{N}_\tau(w) = \cap\{£:£ \in \tau(w)\}$.We said a set $N \in \mathcal{N}_\tau(w)$ a neighborhood of w.
Definition 7[1]: The system of filters $\mathcal{F}(\Xi)$ is partially ordered $£_1 \le £_2$ iff $£_1 \supseteq £_2$. If $\tau_1 \ge \tau_2$, then $\tau_1(w) \subseteq \tau_2(w) \forall w \in \Xi_1$.

Definition 8[6]: (Ξ_1,τ_1) and (Ξ_2,τ_2) are $T_{\tilde{p}VS}$ and f: $\Xi_1 \to \Xi_2$ a map. The map f: (Ξ_1,τ_1)→(Ξ_2,τ_2) is continuous at a point w ∈ Ξ_1 if ∀ filter £ ∈ τ_1(w), f(£) ∈ $\tau_2(f(w))$.

Definition 9[6]: Let $A \subseteq \Xi$. τ_A is subspace pseudo structure on A , and it is the initial pseudo structure with respect to the inclusion mapping $in: A \to \Xi$. Let £ ∈ Ƒ (A) (Ƒ (A) is the set of all the filters that defined of A)and $w \in A$. Then £ ∈ τ_A (w) iff $[£]_\Xi \in \tau$ (w).

Theorem 10[6]: Let f: (Ξ_1,τ_1)→ (Ξ_2,τ_2) be continuous and let (Ξ, $\tilde{\tau}_1$) and (Ξ, $\tilde{\tau}_2$) be other $T_{\tilde{p}VS}$ such that $\tilde{\tau}_1 \geq \tau_1$ and $\tilde{\tau}_2 \leq \tau_2$, then, f: (Ξ_1,$\tilde{\tau}_1$)→ (Ξ_2,$\tilde{\tau}_2$) is also continuous.

Theorem 11[6]: Let $A \subseteq \Xi$, then $\overline{B}^{\tau_A} = \overline{B^{\tau} \cap A}$ ∀ $B \subseteq A$.

Definition 12[6]: Define τ_T as: ∀ $w \in \Xi$, £ ∈ τ_T (w) if and only if £ ≤ $\mathrm{N}_\tau(w)$.

The τ_T is called the natural pseudo structure of the given topology T.

1. Main results

Definition 1: A $T_{\tilde{p}VS}$ (Ξ, τ) is t-connected (Or briefly connected) if the only continuous functions from (Ξ, τ) onto a discrete topological space are constant functions we denote the discrete topological space by (Ξ,T).

Definition 2: If (Ξ, τ) is a $T_{\tilde{p}VS}$and $A \subseteq \Xi$. Then A is a t-connected subset of Ξ if (A , t_A) is a connected $T_{\tilde{p}VS}$.

Theorem 3.3: Let (Ξ, τ_1) and (Ξ, τ_2) be $T_{\tilde{p}VS}$ with $\tau_1 \geq \tau_2$. If (Ξ, τ_1) is connected, then (Ξ, τ_2)is connected. Furthermore, if A is a τ_1-connected subset of Ξ, then A is a τ_2-connected subset of Ξ.

Proof: To prove the first assertion, assume that $f: (\Xi, \tau_2) \to (\Xi, T)$is a continuous function, then $f: (\Xi, \tau_1) \to (\Xi, T)$is continuous by Theorem 2.10 and $\tau_1 \geq \tau_2$. Hence f is constant as (Ξ, τ_1) is connected. This means that (Ξ, τ_2) is a connected $T_{\tilde{p}VS}$.The second assertion follows, because $\tau_A \geq \tau_A$ whenever $\tau_1 \geq \tau_2$.

Theorem 4: Let (Ξ, τ) be a $T_{\tilde{p}VS}$. Then, the sentences are equivalent:

(1) (Ξ, τ) is connected. (2) (Ξ, τ_T) is connected.

Proof: (1) is equivalent (2) Since a function $f: (\Xi, \tau) \to (\Xi, T)$ is continuous
if and only if $f: (\Xi, \tau_T) \to (\Xi, T)$ is continuous, we get that (Ξ, τ) is connected if and only if (Ξ, τ_T) is connected.

Theorem 5: Let $f: (\Xi_1, \tau_1) \to (\Xi_2, \tau_2)$ be a continuous mapping from the connected $T_{\tilde{p}VS}$ (Ξ_1, τ_1) onto the $T_{\tilde{p}VS}$ (Ξ_2, τ_2). Then, (Ξ_2, τ_2) is connected.

Proof: Let h: $(\Xi_2, \tau_2) \to (\Xi_3, T)$ be a continuous map, then $h \circ f$: $(\Xi_1, \tau_1) \to (\Xi_3, T)$ is continuous, and since f is onto we get that h is a constant function. Hence, (Ξ_2, τ_2)is connected.

Definition 6: Two nonempty and proper subsets A and B of Ξ are separated in
(Ξ, τ)if $\bar{A} \cap B = A \cap \bar{B} = \varphi$

Theorem 7: Let (Ξ, τ) be a $T_{\tilde{p}VS}$. Then, the statements are equivalent:

(a) (Ξ, τ) is connected. (b) There is no proper subset of Ξ that is both open and closed. (c) Ξ cannot be represented as the union of two disjoint open sets.

d) Ξ cannot be represented as the union of two disjoint closed sets.

e) Ξ cannot be represented as the union of two separated sets.

Proof: Follows by Theorem 3.4 and theorems from topology.

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Theorem 8: Let (Ξ, τ) be a $T_{\tilde{p}VS}$ and $A \subseteq \Xi$, then A is connected if and only if A can not be written as the union of two separated sets in Ξ, each of which has a nonempty intersection with A.

Proof: Assume that A is connected and $A = D \cup B$ where $A \cap D \neq \varphi$ and $A \cap B \neq \varphi$ and $\overline{D^{\tau}} \cap B = D \cap \overline{B^{\tau}} = \varphi$. $\overline{D^{\tau_A}} \cap B = D \cap \overline{B^{\tau_A}} = \varphi$ by Theorem 3.11. But B and D are complementary sets relative to A. Hence, $\overline{D^{\tau_A}} \subseteq D$ and $\overline{B^{\tau_A}} \subseteq B$. Hence D and B are τ_A-closed and τ_A-open which is a contradiction as A is connected if and only if (A, τ_A) is connected if and only if there is no proper subsets of A that is both τ_A-open and τ_A-closed by Theorem 3.7.

Conversely, assume that A is not connected. Hence there is a nonempty proper subset B of A which is τ_A-open and τ_A-closed by Theorem 3.7 and Definition 3.2. Hence, $\overline{B^{\tau_A}} = B$ and $\overline{(A/B)^{\tau_A}} = A / B$. Thus $\overline{B^{\tau}} \cap (A/B) = \overline{(A / B)^{\tau}} \cap B = \varphi$. Hence, B and A / B are separated sets in Ξ each of which has a nonempty intersection with A which is a contradiction of the assumption. Therefore, A is a connected.

Theorem 9: Let A be a connected subset of Ξ and $B \subseteq \Xi$ such that $A \subset B \subset \bar{A}$. Then, B is connected.

Proof: Assume that that $A \subset B \subset \bar{A}$ and A is connected.

If B is not connected, then there is a discrete space (Ξ, T) and a function f from B onto Ξ which is continuous with respect to τ_B and T and which is not constant. Let, f_A be the restriction of f to A, then f_A:(A, τ_A)) $\rightarrow (f(A), T_{f(A)})$ is continuous. The function f_A is constant because A is connected and $(f(A), T_{f(A)})$ is a discrete space. Let $f_A = \{e\}$. The function f is not constant, so there is a $b \in B$ such that $f(b) \neq e$. The point $b \in \bar{A}$; therefore, there is $£ \in \tau(b)$ such that $A \in £$. Let $\wp = \{B \cap F: F \in £\}$.The collection $\wp$ is a filter base because $A \in £$ and $A \subseteq B$. The filter generated by $\wp$ on $\Xi = £$. Hence, the filter generated by $\wp$ on B, τ_B-converges to b. Let $\mathcal{H}$ denote the filter generated by $\wp$ on B. $f(\mathcal{H}) = [e]$, as the set $A \in \mathcal{H}$. But $\mathcal{H}_{\tau_B}$ -converges to b and $f(b) \neq e$ and the topology T is discrete, therefore $f(\mathcal{H})$ does not converge to $f(b)$. This contradicts the continuity of f. Therefore, B is connected.

Theorem 10: Let (Ξ, τ) be a $T_{\tilde{p}VS}$ and $A \subseteq \Xi$. Then:

(1) If A is connected then A is τ_T-connected. (2) If $T(\tau_A) = [\tau_T]A$ and A is τ_T–connected then A is connected.

Proof: (1) Since $\tau \geq \tau_T$we get by Theorem 3.3 that if A is connected, then A is τ_T -connected. (2) If $[\tau_T]A = T(\tau_A)$ and A is τ_T-connected then $(A, [\tau_T]A)$ is connected, Hence $(A, t(\rho_A))$ is connected. Therefore, (A, ρ_A) is connected by Theorem 3.4. Consequently, A is connected by Definition 3.2

Theorem 11: Let (Ξ, τ) be a $T_{\tilde{p}VS}$and $A \subseteq \Xi$. Then, if A is connected, then $\overline{A^{\tau_T}}$ is τ -connected**.**

Proof: If A is connected then A is τ_T-connected by Theorem 3.3 which implies that $\overline{A^{\tau_T}}$ is τ_T-connected by Theorem 3.9 and since $\overline{A^{\tau_T}}$ is closed we have $\overline{A^{\tau_T}}$ is τ-connected by Theorem 3.10(b).

The following example and Theorem 3.3 show that in general the set of connected subsets of a $T_{\tilde{p}VS}$ may be strictly subset of the set of connected subsets of its topological modification space. we can have $A \subset B \subset \overline{A^{\tau_T}}$ and A is connected but B is not connected.

Example 12: Let (Ξ, τ) be a $T_{\tilde{p}VS}$, and $\Xi = \{x_n : n \in Z\}$, the neighbourhood filters defined as follows: for each $n \in Z, \mathcal{U}_{\tau}(x_n)$ is the filter generated by$\{x_n - 1, x_n, x_n + 1\}$.(1)The topology is indiscrete.

(2) Let $A = \{ x_n : n$ is an even integer$\}$. Then, τ_A is the discrete topology on A.

Proof: (1)Let A be a nonempty open subset of Ξ such that Then there exist such that x and $x_i - 1\ or\ x_i + 1 \in \Xi \setminus A$. But A is open then, $A \in £, \forall £ \geq \mathcal{U}(x_i)$. This implies that $\{x_i - 1, x_i, x_i + 1\} \subseteq A$ which is a contradiction as x_i-1 or $x_i + 1 \in \Xi \setminus A$. Hence, A cannot be open. Therefore, τ_T is indiscrete topology.

(2) Let $xi \in A$ then $£ \in \tau_T(xi)$ if and only if $[£]_\Xi \in \tau(x_i)$.

Since $A, \{xi - 1, xi, xi + 1\} \in [£]_\Xi$, we have, $A \cap \{xi - 1, xi, xi + 1\} = \{x_i\} \in [£]_\Xi$, and this means $[£]_\Xi = [x_i]$ on Ξ. Hence, $£ = [x_i]$ on A . Hence (A, τ_A) is the discrete topology. $T(\tau_A) = \tau_A$since τ_A is the discrete topology on A. The topology$[\tau_T]A$ is the indiscrete topology on A. So $[\tau_T]A < T(\tau_A)$ and $(A, [\tau_T]A)$ is connected but $(A, T(\tau_A))$ is not connected. Hence, A is τ_T-connected subset of Ξ but not a connected subset of Ξ. If $B \subseteq A$ where B is the set containing one element. Then B is connected also $\overline{B^{\tau_T}} = \Xi$ is connected by Theorem 3.11 and since τ_T is an indiscrete topology. But $B \subset A \subset \Xi = \overline{B^{\tau_T}}$. But A is not connected.

Theorem 13: If $H\ and\ K$ are separated in the $T_{\tilde{p}VS}$ (Ξ, τ) and F is a connected sub sets of $H \cup K$. Then, $F \subseteq H$or $F \subseteq K$.

Proof: Assume that $F \subseteq H \cup K$ and $F \nsubseteq H$ and $F \nsubseteq K$ then $F \cap H$and $F \cap K$ are separated in (Ξ, τ). Since $\overline{F \cap H} \cap (F \cap H) \subseteq \bar{F} \cap \bar{H} \cap F \cap H = \varphi$. And $(F \cap H) \cap \overline{F \cap K} \subseteq (F \cap H) \cap \bar{F} \cap \bar{K} = \varphi$. as $H, K$are separated. But $(F \cap H) \cup (F \cap K) = F$ so F is not connected by Theorem 3.8 which is a contradiction as F is connected. Hence, $F \subseteq H$ or $F \subseteq K$.

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