

## Existence and Uniqueness of Anti-Periodic Solution for Second Order Impulsive Differential Equations

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### Abstract:

In this article, we study the existence and uniqueness of anti-periodic solution to a second order impulsive differential equation as follows:

$$\begin{aligned}x''(t) + \phi(t)f(t, x(t), x'(t)) &= e(t) \text{ for } t \neq t_k, k \in \mathbb{Z} \\ \Delta x(t_k) &= x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \\ \Delta x'(t_k) &= x'(t_k^+) - x'(t_k^-) = \bar{I}_k(x'(t_k^-)),\end{aligned}$$

where  $f \in C(\mathbb{R}^3, \mathbb{R})$ ,  $\phi \in C(\mathbb{R}, \mathbb{R})$ ,  $e, I_k, \bar{I}_k \in C(\mathbb{R}, \mathbb{R})$ ,  $f$  in the first argument and  $\phi$  are  $\omega$  periodic,  $\omega$  is a positive constant. By using the method of coincidence degree.

**Keywords:** Impulsive differential equations, Existence, Anti-periodic solution, Coincidence degree.

## 1. Introduction

In mathematical terms, when we describe the evolution of a real process impacted by perturbations in the short term, it can be convenient to neglect the duration of these perturbations and treat them as if they were occurring instantly, i.e. in the form of impulses. Thus, impulsive differential equations naturally play a role in modeling many evolutionary phenomena observed in the real world, such as rupture rate models in medicine and biology, optimal control models in economics, pharmacokinetics, and frequency modulation systems [6]. Over the years, researchers have conducted many studies addressing the problem of establishing the existence of solutions for various types of impulsive differential equations, for some results and applications of the impulsive differential equations see [2,7,8].

Yongkun, Tianwei [7] studied the following problem has a unique  $\frac{\omega}{2}$ -anti-periodic solution by using the method of coincidence degree,

$$\begin{cases}x''(t) + f(t, x'(t)) + g(t, x(t), x(t - \tau_{(t,x)})) = p(t), t \neq t_k, k \in \mathbb{Z}, \\ x(t_k^+) = x(t_k^-) + I_k(x(t_k^-)), x'(t_k^+) = x'(t_k^-) + L_k(x(t_k^-), x'(t_k^-)),\end{cases}$$

in this problem,  $f$  depends only on  $t$ , and  $x'(t)$ , and  $g$  depends on  $t, x(t)$ , and  $\tau_{(t,x)}$ , where  $\tau_{(t,x)}$  is a time and state-dependent delay.

Motivated by the work mentioned above, we study the existence of anti-periodic solutions of the more general second order impulsive differential equations:

$$\begin{aligned} x''(t) + \phi(t)f(t, x(t), x'(t)) &= e(t) \text{ for } t \neq t_k, k \in \mathbb{Z} \\ \Delta x(t_k) &= x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \\ \Delta x'(t_k) &= x'(t_k^+) - x'(t_k^-) = \bar{I}_k(x'(t_k^-)), \end{aligned} \quad (1.1)$$

where  $f \in C(\mathbb{R}^3, \mathbb{R})$  and  $\phi \in C(\mathbb{R}, \mathbb{R})$  are given functions,  $e, I_k, \bar{I}_k \in C(\mathbb{R}, \mathbb{R})$ ,  $f$  in the first argument and  $\phi$  are  $\omega$  periodic,  $\omega$  is a positive constant. There exists a positive integer  $q$  such that  $t_{k+q} = t_k + \frac{\omega}{2}$ ,  $I_{k+q}(u) = -I_k(-u)$ ,  $\bar{I}_{k+q}(u) = -\bar{I}_k(-u)$  for all  $u, v \in \mathbb{R}, k \in \mathbb{Z}$ . We also assume that  $[0, \frac{\omega}{2}) \cap \{t_k : k \in \mathbb{Z}\} = \{t_1, t_2, \dots, t_q\}$ .

## 2 Preliminaries

For the remainder of this section, we present some results which will be needed in section 3. We use a fixed point theorem of coincidence degree to establish an existence of solution for system (1.1).

In the following, we introduce first some notations.

Let

$PC^n([0, \omega], \mathbb{R}) = \{x: [0, \omega] \rightarrow \mathbb{R} \mid x^{(n)}(t), n = 0, 1, 2, \text{ is a piecewise continuous map with first-class discontinuous points in } [0, \omega] \cap \{t_k : k \in \mathbb{Z}\} \text{ and at each discontinuous point it is continuous on the left}\}$ . We set

$$\mathcal{X} = \left\{ x \in PC^1([0, \omega], \mathbb{R}) : x\left(t + \frac{\omega}{2}\right) = -x(t) \text{ for all } t \in \left[0, \frac{\omega}{2}\right] \right\}$$

and

$$\mathbb{Y} = \mathcal{X} \times \mathbb{R}^{2q}$$

be two Banach spaces with the norms

$$|x|_{\mathcal{X}} = \max\{|x|_{\infty}, |x'|_{\infty}\} \text{ and } |z|_{\mathbb{Y}} = |x|_{\mathcal{X}} + |y| \text{ for all } x \in \mathcal{X}, z \in \mathbb{Y}, y \in \mathbb{R}^{2q}$$

in which  $|x|_{\infty} = \max_{t \in [0, \frac{\omega}{2}]} |x(t)|$ ,  $|x'|_{\infty} = \max_{t \in [0, \frac{\omega}{2}]} |x'(t)|$ ,  $|\cdot|$  is any norm of  $\mathbb{R}^{2q}$ .

**Definition 2.1** ([4]). Let  $X$  and  $Y$  be normed vector spaces and consider the linear mapping  $L : D(L) \subset X \rightarrow Y$ .  $L$  is called a Fredholm mapping of index 0 if  $\dim(\text{Ker}(L)) = \text{codim}(\text{Im}(L)) < \infty$  and  $\text{Im}(L)$  is a closed subset of  $Y$ .

**Remark 2.1** ([4]). If  $L$  is a Fredholm mapping of index 0, then there exist continuous projectors  $P: X \rightarrow X, Q: Y \rightarrow Y$  such that  $\text{Im}(P) = \text{Ker}(L)$  and  $\text{Im}(L) = \text{Ker}(Q) = \text{Im}(I - Q)$ .

It follows that  $L|_{D(L) \cap \text{Ker}(P)}: (I - P)X \rightarrow \text{Im}(L)$  is an invertible map, with inverse  $K_P$ .

**Definition 2.2**([4]). Let  $N: X \rightarrow Y$  be a continuous map between two normed spaces and  $\Omega$  an open bounded subset of  $X$ . We say that  $N$  is  $L$ -compact on  $\bar{\Omega}$  if  $QN(\bar{\Omega})$  is bounded and  $K_P(I - Q)N: \bar{\Omega} \rightarrow X$  is a compact map.

**Lemma 2.1** ([4], [7]). Let  $\mathcal{X}, \mathcal{Y}$  be two Banach spaces,  $\Omega \subset \mathcal{X}$  be open bounded and symmetric with  $0 \in \Omega$ . Suppose that  $L: D(L) \subset \mathcal{X} \rightarrow \mathcal{Y}$  is a linear Fredholm operator of index zero with  $D(L) \cap \bar{\Omega} \neq \emptyset$  and  $N: \bar{\Omega} \rightarrow \mathcal{Y}$  is  $L$ -compact. Further, we also assume that

$$Lx - Nx \neq \lambda(-Lx - N(-x))$$

for all  $x \in D(L) \cap \partial\Omega, \lambda \in (0,1]$ .

Then equation  $Lx = Nx$  has at least one solution on  $\text{Dom}L \cap \bar{\Omega}$ .

**Lemma 2.2.** Assume that  $x(t)$  is an  $\frac{\omega}{2}$ -anti-periodic function,  $h \in C(\mathbb{R}, \mathbb{R})$ . If  $x(t)$  is a solution of the following system

$$\begin{cases} x''(t) = h(t), t \neq t_k, k \in \mathbb{Z}, \\ x(t_k^+) = x(t_k^-) + I_k(x(t_k)), x'(t_k^+) = x'(t_k^-) + \bar{I}_k(x'(t_k)), \end{cases} \quad (2.1)$$

then  $x$  satisfies

$$\begin{cases} x'(t) = -\frac{1}{2} \int_t^{t+\frac{\omega}{2}} h(s) ds - \frac{1}{2} \sum_{t_k \in (t, t+\frac{\omega}{2})} \bar{I}_k(x'(t_k)), t \neq t_k, k \in \mathbb{Z}, \\ x(t_k^+) = x(t_k^-) + I_k(x(t_k)). \end{cases}$$

**Proof.** Since  $x(t)$  is an  $\frac{\omega}{2}$ -anti-periodic solution of system (2.1), then by integrating  $x''(t) = h(t), t \neq t_k (k \in \mathbb{Z})$  step by step from  $t$  to  $t + \frac{\omega}{2}$ , we have

$$x'(t + \frac{\omega}{2}) = x'(t) + \int_t^{t+\frac{\omega}{2}} h(s) ds + \sum_{t_k \in (t, t+\frac{\omega}{2})} [x'(t_k^+) - x'(t_k^-)]. \quad (2.2)$$

By  $x'(t + \frac{\omega}{2}) = -x'(t)$  and  $x'(t_k^+) - x'(t_k^-) = \bar{I}_k(x'(t_k)) (k \in \mathbb{Z})$ , it follows from (2.2) that

$$x'(t) = -\frac{1}{2} \int_t^{t+\frac{\omega}{2}} h(s) ds - \frac{1}{2} \sum_{t_k \in (t, t+\frac{\omega}{2})} \bar{I}_k(x'(t_k)).$$

**Lemma 2.3.** Assume that there exist constants  $\alpha, \beta, a_k$  and  $b_k$  such that

( $H_1$ ) For all  $t, x, y, u, v \in \mathbb{R}$ ,

$$|f(t, x, y) - f(t, u, v)| \leq \alpha|x - u| + \beta|y - v|.$$

(H<sub>2</sub>) for all  $t, x, y \in \mathbb{R}$ ,

$$|I_k(x) - I_k(y)| \leq a_k|x - y| \text{ and } |\bar{I}_k(x) - \bar{I}_k(y)| \leq b_k|x - y|.$$

(H<sub>3</sub>)  $1 - \max\{\Lambda_1, \Lambda_2\} > 0$ , where

$$\Lambda_1 = \frac{\omega}{2} \max_{s \in [0, \frac{\omega}{2}]} |\phi(s)|(\alpha + \beta) + \sum_{k=1}^q b_k$$

and

$$\Lambda_2 = \frac{\omega^2}{8} \max_{s \in [0, \frac{\omega}{2}]} |\phi(s)|(\alpha + \beta) + \frac{\omega}{4} \sum_{k=1}^q b_k + \sum_{k=1}^q a_k.$$

Then the set of all possible  $\frac{\omega}{2}$ -anti-periodic solutions of problem

$$\begin{cases} x''(t) = \frac{1}{1+\lambda} G(t, x) - \frac{\lambda}{1+\lambda} G(t, -x), \quad t \neq t_k, \quad k \in \mathbb{Z}, \\ x(t_k^+) = x(t_k^-) + \frac{1}{1+\lambda} I_k(x(t_k)) - \frac{\lambda}{1+\lambda} I_k(-x(t_k)), \\ x'(t_k^+) = x'(t_k^-) + \frac{1}{1+\lambda} \bar{I}_k(x'(t_k)) - \frac{\lambda}{1+\lambda} \bar{I}_k(-x'(t_k)), \end{cases} \quad (2.3)$$

where

$$G(t, x) = e(t) - \phi(t)f(t, x(t), x'(t))$$

and

$$G(t, -x) = e(t) - \phi(t)f(t, -x(t), -x'(t))$$

are bounded in  $\mathcal{X}$ .

**Proof.** Let  $S \subset \mathcal{X}$  be the set of all possible  $\frac{\omega}{2}$ -anti-periodic solutions of (2.3). If  $S = \emptyset$ , the proof is ended. Suppose  $S \neq \emptyset$ , and let  $x \in S$ . From Lemma 2, system (2.3) can reduce to

$$\begin{cases} x'(t) = \frac{1}{2} \int_t^{t+\frac{\omega}{2}} \left[ \frac{\lambda}{1+\lambda} G(s, -x) - \frac{1}{1+\lambda} G(s, x) \right] ds + \frac{1}{2} \sum_{t_k \in (t, t+\frac{\omega}{2})} \left[ \frac{\lambda}{1+\lambda} \bar{I}_k(-x'(t_k)) \right. \\ \left. - \frac{1}{1+\lambda} \bar{I}_k(x'(t_k)) \right], \quad t \neq t_k, \quad k \in \mathbb{Z}, \\ x(t_k^+) = x(t_k^-) + \frac{1}{1+\lambda} I_k(x(t_k)) - \frac{\lambda}{1+\lambda} I_k(-x(t_k)). \end{cases} \quad (2.4)$$

Let  $t_0 = t_0^+ = 0, t_{q+1} = \frac{\omega}{2}$ , in view of (2.3), we get from  $(H_1) - (H_2)$

$$\begin{aligned} \int_0^{\frac{\omega}{2}} |x''(s)| ds &= \sum_{k=1}^{q+1} \int_{t_{k-1}^+}^{t_k} |x''(s)| ds + \sum_{k=1}^q |x'(t_k^+) - x'(t_k^-)| \\ &\leq \int_0^{\frac{\omega}{2}} \left| \frac{1}{1+\lambda} G(s, x) - \frac{\lambda}{1+\lambda} G(s, -x) \right| ds + \sum_{k=1}^q \left| \frac{1}{1+\lambda} \bar{I}_k(x'(t_k)) - \frac{\lambda}{1+\lambda} \bar{I}_k(-x'(t_k)) \right| \\ &\leq \left[ \frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda} \right] \int_0^{\frac{\omega}{2}} \max\{|G(s, x)|, |G(s, -x)|\} ds \\ &\quad + \left[ \frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda} \right] \sum_{k=1}^q \max\{|\bar{I}_k(x'(t_k))|, |\bar{I}_k(-x'(t_k))|\} \\ &\leq \sum_{k=1}^q \max\{|G(s, x) - G(s, 0)|, |G(s, -x) - G(s, 0)|\} ds \\ &\quad + \max_{s \in [0, \frac{\omega}{2}]} |G(s, 0)| \frac{\omega}{2} + \sum_{k=1}^q |\bar{I}_k(0)| \\ &\leq \int_0^{\frac{\omega}{2}} \max\{|-\phi(s)f(s, x(s), x'(s)) + \phi(s)f(s, 0, 0)|, |-\phi(s)f(s, -x(s), -x'(s)) \\ &\quad + \phi(s)f(s, 0, 0)|\} ds + \sum_{k=1}^q b_k |x'(t_k)| + \max_{s \in [0, \frac{\omega}{2}]} |G(s, 0)| \frac{\omega}{2} + \sum_{k=1}^q |\bar{I}_k(0)| \\ &\leq \max_{s \in [0, \frac{\omega}{2}]} |\phi(s)| \int_0^{\frac{\omega}{2}} \max\{|-f(s, x(s), x'(s)) + f(s, 0, 0)|, | -f(s, -x(s), -x'(s)) + f(s, 0, 0)|\} ds \\ &\quad + \sum_{k=1}^q b_k |x'(t_k)| + \max_{s \in [0, \frac{\omega}{2}]} |G(s, 0)| \frac{\omega}{2} + \sum_{k=1}^q |\bar{I}_k(0)| \\ &\leq \frac{\omega}{2} \max_{s \in [0, \frac{\omega}{2}]} |\phi(s)| [\alpha |x|_\infty + \beta |x'|_\infty] + \sum_{k=1}^q b_k |x'(t_k)| + \max_{s \in [0, \frac{\omega}{2}]} |G(s, 0)| \frac{\omega}{2} + \sum_{k=1}^q |\bar{I}_k(0)| \\ &\leq \left[ \frac{\omega}{2} \max_{s \in [0, \frac{\omega}{2}]} |\phi(s)| \beta + \sum_{k=1}^q b_k \right] |x'|_\infty + \frac{\omega}{2} \max_{s \in [0, \frac{\omega}{2}]} |\phi(s)| \alpha |x|_\infty \\ &\quad + \max_{s \in [0, \frac{\omega}{2}]} |G(s, 0)| \frac{\omega}{2} + \sum_{k=1}^q |\bar{I}_k(0)| \\ &\leq \left[ \frac{\omega}{2} \max_{s \in [0, \frac{\omega}{2}]} |\phi(s)| (\alpha + \beta) + \sum_{k=1}^q b_k \right] |x|_X + \max_{s \in [0, \frac{\omega}{2}]} |G(s, 0)| \frac{\omega}{2} + \sum_{k=1}^q |\bar{I}_k(0)| \end{aligned}$$

$$= \Lambda_1 |x|_X + \Theta_1, \tag{2.5}$$

where

$$\Theta_1 = \max_{s \in [0, \frac{\omega}{2}]} |G(s, 0)| \frac{\omega}{2} + \sum_{k=1}^q |\bar{I}_k(0)|.$$

Furthermore, in view of (2.4), we have

$$\begin{aligned} \int_0^{\frac{\omega}{2}} |x'(s)| ds &= \sum_{k=1}^{q+1} \int_{t_{k-1}^+}^{t_k} |x'(s)| ds + \sum_{k=1}^q |x(t_k^+) - x(t_k^-)| \\ &\leq \frac{1}{2} \int_0^{\frac{\omega}{2}} \int_t^{t+\frac{\omega}{2}} \left| \frac{\lambda}{1+\lambda} G(s, -x) - \frac{1}{1+\lambda} G(s, x) \right| ds dt \\ &\quad + \frac{1}{2} \int_0^{\frac{\omega}{2}} \sum_{t_k \in (t, t+\frac{\omega}{2})} \left| \frac{\lambda}{1+\lambda} \bar{I}_k(-x'(t_k)) - \frac{1}{1+\lambda} \bar{I}_k(x'(t_k)) \right| dt \\ &\quad + \sum_{k=1}^q \left| \frac{1}{1+\lambda} I_k(x(t_k)) - \frac{\lambda}{1+\lambda} I_k(-x(t_k)) \right| \\ &\leq \left[ \frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda} \right] \left[ \frac{1}{2} \int_0^{\frac{\omega}{2}} \int_t^{\frac{\omega}{2}} \max\{|G(s, x)|, |G(s, -x)|\} ds dt \right. \\ &\quad \left. + \frac{1}{2} \int_0^{\frac{\omega}{2}} \sum_{k=1}^q \max\{|\bar{I}_k(x'(t_k))|, |I_k(-x'(t_k))|\} dt + \sum_{k=1}^q \max\{|I_k(x(t_k))|, |I_k(-x(t_k))|\} \right] \\ &\leq \frac{1}{2} \int_0^{\frac{\omega}{2}} \int_t^{t+\frac{\omega}{2}} \max\{|G(s, x) - G(s, 0)|, |G(s, -x) - G(s, 0)|\} ds dt \\ &\quad + \frac{\omega}{4} \sum_{k=1}^q \max\{|\bar{I}_k(x'(t_k)) - \bar{I}_k(0)|, |\bar{I}_k(-x'(t_k)) - \bar{I}_k(0)|\} \\ &\quad + \sum_{k=1}^q \max\{|I_k(x(t_k)) - I_k(0)|, |I_k(-x(t_k)) - I_k(0)|\} \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \int_0^{\frac{\omega}{2}} \int_t^{t+\frac{\omega}{2}} |G(s, 0)| ds dt + \frac{\omega}{4} \sum_{k=1}^q |\bar{I}_k(0)| + \sum_{k=1}^q |I_k(0)| \\
 & \leq \frac{1}{2} \int_0^{\frac{\omega}{2}} \int_t^{t+\frac{\omega}{2}} \max\{|-\phi(s)f(s, x(s), x'(s)) + \phi(s)f(s, 0, 0)|, \\
 & \quad |-\phi(s)f(s, -x(s), -x'(s)) + \phi(s)f(s, 0, 0)|\} ds dt \\
 & + \frac{\omega}{4} \sum_{k=1}^q b_k |x'(t_k)| + \sum_{k=1}^q a_k |x(t_k)| + \frac{\omega^2}{8} \max_{s \in [0, \frac{\omega}{2}]} |G(s, 0)| + \frac{\omega}{4} \sum_{k=1}^q |\bar{I}_k(0)| + \sum_{k=1}^q |I_k(0)| \\
 & \leq \left[ \frac{\omega^2}{8} \max_{s \in [0, \frac{\omega}{2}]} |\phi(s)|\alpha + \sum_{k=1}^q a_k \right] |x|_\infty + \left[ \frac{\omega}{4} \sum_{k=1}^q b_k + \frac{\omega^2}{8} \max_{s \in [0, \frac{\omega}{2}]} |\phi(s)|\beta \right] |x'|_\infty \\
 & + \frac{\omega^2}{8} \max_{s \in [0, \frac{\omega}{2}]} |G(s, 0)| + \frac{\omega}{4} \sum_{k=1}^q |\bar{I}_k(0)| + \sum_{k=1}^q |I_k(0)| \\
 & \leq \left[ \frac{\omega^2}{8} \max_{s \in [0, \frac{\omega}{2}]} |\phi(s)|(\alpha + \beta) + \frac{\omega}{4} \sum_{k=1}^q b_k + \sum_{k=1}^q a_k \right] |x|_X + \frac{\omega^2}{8} \max_{s \in [0, \frac{\omega}{2}]} |G(s, 0)| \\
 & + \frac{\omega}{4} \sum_{k=1}^q |\bar{I}_k(0)| + \sum_{k=1}^q |I_k(0)| \\
 & = \Lambda_2 |x|_X + \Theta_2, \tag{2.6}
 \end{aligned}$$

where

$$\Theta_2 = \frac{\omega^2}{8} \max_{s \in [0, \frac{\omega}{2}]} |G(s, 0)| + \frac{\omega}{4} \sum_{k=1}^q |\bar{I}_k(0)| + \sum_{k=1}^q |I_k(0)|.$$

Since  $x(t)$  is an arbitrary  $\frac{\omega}{2}$ -anti-periodic solution of system (2.3), so  $x'(t)$  is  $\frac{\omega}{2}$ -anti-periodic.

Hence, there exist constants  $u, v \in [0, \frac{\omega}{2}]$  such that

$$x(u) = x(u^-) = 0 \text{ and } x'(v) = x'(v^-) = 0.$$

Therefore, for  $t \in [0, \frac{\omega}{2}]$ , we have

$$|x'(t)| \leq |x'(v)| + \int_0^{\frac{\omega}{2}} |x''(s)| ds = \int_0^{\frac{\omega}{2}} |x''(s)| ds \tag{2.7}$$

and

$$|x(t)| \leq |x(u)| + \int_0^{\frac{\omega}{2}} |x'(s)| ds = \int_0^{\frac{\omega}{2}} |x'(s)| ds \tag{2.8}$$

Then we obtain from (2.5) -(2.8) that

$$|x'|_{\infty} = \max_{t \in [0, \frac{\omega}{2}]} |x'(t)| \leq \int_0^{\frac{\omega}{2}} |x''(s)| ds \leq \Lambda_1 |x|_X + \Theta_1$$

and

$$|x|_{\infty} = \max_{t \in [0, \frac{\omega}{2}]} |x(t)| \leq \int_0^{\frac{\omega}{2}} |x'(s)| ds \leq \Lambda_2 |x|_X + \Theta_2.$$

So

$$|x|_X = \max\{|x|_{\infty}, |x'|_{\infty}\} \leq \max\{\Lambda_1, \Lambda_2\} |x|_X + \max\{\Theta_1, \Theta_2\}$$

which implies from (H<sub>3</sub>) that

$$|x|_X \leq \frac{\max\{\Theta_1, \Theta_2\}}{1 - \max\{\Lambda_1, \Lambda_2\}}.$$

### 3 Existence

Theorem 3.1. Assume that  $\phi$  is a  $\frac{\omega}{2}$  periodic function. In addition, we suppose that there exist constants  $\alpha, \beta, a_k$  and  $b_k$  such that the following conditions are satisfied:

(H<sub>1</sub>) For all  $t, x, y, u, v \in \mathbb{R}$ ,

$$|f(t, x, y) - f(t, u, v)| \leq \alpha|x - u| + \beta|y - v|.$$

(H<sub>2</sub>) for all  $t, x, y \in \mathbb{R}$  and  $\forall k \in \mathbb{Z}$ ,

$$|I_k(x) - I_k(y)| \leq a_k|x - y| \text{ and } |\bar{I}_k(x) - \bar{I}_k(y)| \leq b_k|x - y|.$$

(H<sub>3</sub>) For all  $t, x, y \in \mathbb{R}$ ,

$$f\left(t + \frac{\omega}{2}, -x, -y\right) = -f(t, x, y), \quad e\left(t + \frac{\omega}{2}\right) = -e(t)$$

(H<sub>4</sub>)  $1 - \max\{\Lambda_1, \Lambda_2\} > 0$ , where

$$\Lambda_1 = \frac{\omega}{2} \max_{s \in [0, \frac{\omega}{2}]} |\phi(s)|(\alpha + \beta) + \sum_{k=1}^q b_k$$

and

$$\Lambda_2 = \frac{\omega^2}{8} \max_{s \in [0, \frac{\omega}{2}]} |\phi(s)| (\alpha + \beta) + \frac{\omega}{4} \sum_{k=1}^q b_k + \sum_{k=1}^q a_k.$$

Then (1.1) has a unique  $\frac{\omega}{2}$ -anti-periodic solution.

Proof. Lemma 1 will be applied to obtaining the existence of  $\frac{\omega}{2}$ -anti-periodic solution of (1.1). By Lemma 3, there exists  $M$  such that, for any  $\frac{\omega}{2}$ -anti-periodic solution  $x(t)$  of (2.3),

$$|x|_X \leq M$$

where

$$M := \frac{\max\{\Theta_1, \Theta_2\}}{1 - \max\{\Lambda_1, \Lambda_2\}} + 1.$$

Set

$$\Omega = \{x \in X : |x|_X < M\} \subset X.$$

Then define a linear operator  $L: D(L) \subset X \rightarrow Y$  by

$$x \rightarrow \left( x'', \Delta x(t_1), \Delta x(t_2), \dots, \Delta x(t_q), \Delta x'(t_1), \Delta x'(t_2), \dots, \Delta x'(t_q) \right),$$

where  $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ ,  $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$ ,  $k \in \mathbb{Z}$ , and

$$D(L) = \left\{ x \in PC^1([0, \omega], \mathbb{R}) : x\left(t + \frac{\omega}{2}\right) = -x(t) \text{ for all } t \in \left[0, \frac{\omega}{2}\right] \right\},$$

$N: X \rightarrow Y$

$$Nx = \left( F(t), I_1(x(t_1)), I_2(x(t_2)), \dots, I_q(x(t_q)), \bar{I}_1(x'(t_1)), \bar{I}_2(x'(t_2)), \dots, \bar{I}_q(x'(t_q)) \right),$$

where

$$F(t) = e(t) - \phi(t)f(t, x(t), x'(t)).$$

It is easy to see that

$$\text{Ker } L = \{0\} \text{ and } \text{Im } L = \left\{ z = (f, c_1, \dots, c_q, d_1, \dots, d_q) \in Y : \int_0^\omega f(s) ds = 0 \right\} \equiv Y.$$

Thus

$$\dim \text{Ker } L = 0 = \text{codim Im } L,$$

and  $L$  is a linear Fredholm operator of index zero.

Define the continuous projectors  $P: \mathcal{X} \rightarrow \mathcal{X}$  by

$$Px = \int_0^\omega x(s)ds = 0,$$

and  $Q: \mathbb{Y} \rightarrow \mathbb{Y}$  by

$$Qz = Q(f, c_1, \dots, c_q, d_1, \dots, d_q) = \left( \int_0^\omega f(s)ds, 0, \dots, 0, 0, \dots, 0 \right).$$

Hence  $\text{Im } P = \text{Ker } L$  and  $\text{Ker } Q = \text{Im } L$ .

Let  $L_P^{-1}: \text{Im } L \rightarrow D(L) \cap \text{Ker } P$  the inverse of  $L|_{D(L) \cap \text{Ker } P}$

$$\begin{aligned} L_P^{-1}z &= \frac{\omega - 4t}{8} \int_0^{\frac{\omega}{2}} f(s)ds + \int_0^t (t - s)f(s)ds - \frac{1}{2} \int_0^{\frac{\omega}{2}} \left( \frac{\omega}{2} - s \right) f(s)ds + \sum_{0 < t_k < t} (t - t_k)d_k \\ &+ \sum_{0 < t_k < t} c_k + \sum_{k=1}^q \left( \frac{t_k}{2} - \frac{\omega}{8} - \frac{t}{2} \right) d_k - \frac{1}{2} \sum_{k=1}^q c_k, \end{aligned}$$

in which  $c_{q+i} = -c_i, d_{q+i} = -d_i$  for all  $1 \leq i \leq q$ .

Clearly,  $QN$  and  $L_P^{-1}(I - Q)N$  are continuous.

Using the Arzela-Ascoli theorem, it is not difficult to show that  $QN(\bar{\Omega})$  is bounded and  $L_P^{-1}(I - Q)N(\bar{\Omega})$  is relatively compact. Therefore,  $N$  is  $L$ -compact on  $\bar{\Omega}$ .

Now conditions of Lemma1 are satisfied because if there exists  $x \in \partial\Omega \cap D(L)$  and  $\lambda \in [0,1]$  such that

$$Lx - Nx = \lambda(-Lx - N(-x))$$

i.e.

$$(1 + \lambda)Lx = Nx - \lambda N(-x)$$

then  $x$  is a  $\frac{\omega}{2}$ -anti-periodic solution of (2.3).

So by Lemma 3 , we have

$$|x|_X \leq \frac{\max\{\Theta_1, \Theta_2\}}{1 - \max\{\Lambda_1, \Lambda_2\}}$$

which implies that

$$|x|_X < M.$$

Since  $x \in \partial\Omega \cap D(L)$ ; so  $|x|_X = M$  a contradiction. Thus,

$$Lx - Nx \neq \lambda(-Lx - N(-x))$$

for  $x \in \partial\Omega \cap D(L)$  and  $\lambda \in [0,1]$ . Then by Lemma1 problem (1.1) has at least one solution on  $DomL \cap \bar{\Omega}$ .

Suppose that  $x(t)$  and  $y(t)$  are two  $\frac{\omega}{2}$ -anti-periodic solutions of system (1.1). Set  $z(t) = x(t) - y(t)$ . Then we have

$$\begin{cases} z''(t) = G(t, z), t \neq t_k, k \in \mathbb{Z} \\ z(t_k^+) = z(t_k^-) + I_k(x(t_k)) - I_k(y(t_k)), \\ z'(t_k^+) = z'(t_k^-) + \bar{I}_k(x'(t_k)) - \bar{I}_k(y'(t_k)), \end{cases} \quad (3.1)$$

where

$$G(t, z) = \phi(t)f(t, y(t), y'(t)) - \phi(t)f(t, x(t), x'(t)).$$

From Lemma 2, system (3.1) can reduce to

$$\begin{cases} z'(t) = -\frac{1}{2} \int_t^{t+\frac{\omega}{2}} G(s, z) ds \\ \quad - \frac{1}{2} \sum_{t_k \in (t, t+\frac{\omega}{2})} [\bar{I}_k(x'(t_k),) - \bar{I}_k(y'(t_k),)], t \neq t_k, k \in \mathbb{Z} \\ z(t_k^+) = z(t_k^-) + I_k(x(t_k)) - I_k(y(t_k)). \end{cases}$$

By using a similar argument as that in the proof of Lemma 2.3, we can obtain that

$$|z'|_\infty = \max_{t \in [0, \frac{\omega}{2}]} |z'(t)| \leq \int_0^{\frac{\omega}{2}} |z''(s)| ds \leq \Lambda_1 \|z\|_X,$$

and

$$|z|_\infty = \max_{t \in [0, \frac{\omega}{2}]} |z(t)| \leq \int_0^{\frac{\omega}{2}} |z'(s)| ds \leq \Lambda_2 |z|_X.$$

So

$$|z|_X = \max\{|z|_\infty, |z'|_\infty\} \leq \max\{\Lambda_1, \Lambda_2\} |z|_X.$$

Since  $z(t)$  and  $z'(t)$  are anti-periodic functions, it follows from (H<sub>4</sub>) that

$$z(t) = z'(t) = 0 \text{ for all } t \in \mathbb{R}.$$

Thus  $x(t) = y(t)$  for all  $t \in \mathbb{R}$ . Therefore, system (1.1) has a unique  $\frac{\omega}{2}$ -anti-periodic solution.

#### 4 Example

In this section, we present an example to demonstrate the application of Theorem 3.1. Let us consider the impulsive differential equations:

$$\begin{aligned}x''(t) + \frac{1}{8\pi^3} (\sin(t))^2 \cos(t) |(x(t) - x'(t))| &= 0 \text{ for } t \neq t_k, k \in \mathbb{Z} \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) &= \frac{1}{\pi^2} x(t_k), \\ \Delta x'(t_k) = x'(t_k^+) - x'(t_k^-) &= \frac{1}{\pi^2} x'(t_k),\end{aligned}\tag{4.1}$$

where  $f(t, x, y) = \frac{1}{2\pi} \cos(t) |(x(t) - y(t))|$ ,  $\phi(t) = \frac{1}{4\pi^2} (\sin(t))^2$ ,  $e(t) = 0$ ,  $I_k(x) = \bar{I}_k(x) = \frac{1}{\pi^2} x$ , and  $[0, \omega] \cap \{t_k : k \in \mathbb{Z}\} = \{t_1, t_2, t_3\}$ .

Thus  $\alpha = \beta = \frac{1}{2\pi}$ , and  $a_k = b_k = \frac{1}{\pi^2}$ .

We have  $\phi$  is  $\pi$ -periodic and we can easily see that assumptions  $(H_1) - (H_4)$  hold. Then by Theorem 3.1, system (4.1) has a unique  $\pi$ -anti-periodic solution.

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